THE MINIMUM b_2 PROBLEM FOR RIGHT-ANGLED ARTIN GROUPS

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Abstract. This paper focuses on tools for constructing 4-manifolds which have fundamental group G isomorphic to a right-angled Artin group and which are also minimal, in the sense that they minimize $b_2(M)$, the dimension of $H_2(M;\mathbb{Q})$. For a finitely presented group G, define $h(G) = \min\{b_2(M)|M \in \mathcal{M}(G)\}$.

In this paper, we explore the ways in which we can bound $h(G)$ from below using group cohomology and the tools necessary to build 4-manifolds that realize these lower bounds. We give solutions for right-angled Artin groups, or RAAGs, when the graph associated to G has no 4-cliques, and further we reduce this problem to the case when the graph is connected and contains only 4-cliques. We then give solutions for many infinite families of RAAGs and provide a conjecture to the solution for all RAAGs.

1. Introduction

It is well known that for any finitely presented group G there is a closed, orientable 4-dimensional manifold with fundamental group isomorphic to G. This paper explores the problem of constructing a 4-manifold M with particular fundamental group that minimizes $b_2(M)$, the dimension of $H_2(M;\mathbb{Q})$. We will refer to this as the *minimum b_2 problem*. Many have researched this topic, including Hausmann and Weinberger [6], Baldridge and Kirk [1,2], Eckmann [5], Johnson and Kotschick [7] and independently Kotschick [10,11], Lučk [12], and most recently Kirk and Livingston [8]. However, the minimum b_2 problem remains open for all but a few classes of groups.

Let $\mathcal{M}(G)$ denote the class of closed, oriented, topological 4-manifolds with fundamental group isomorphic to a fixed group G. For a finitely presented group G, we define

$$h(G) = \min\{b_2(M)|M \in \mathcal{M}(G)\}.$$

Calculations of h are known for free groups and free abelian groups, but little more. This paper generalizes these calculations to right-angled Artin groups, of which free and free abelian groups are special cases. In particular, a right-angled Artin group (abbreviated RAAG) has a presentation with a finite generating set in which the relations consist solely of commutators between generators. RAAGs are also known as graph groups because their presentations can uniquely be represented by graphs, where each vertex represents a generator and each edge between vertices represents a commutator relation between those generators. Hence, F_n is associated to a graph with n vertices with no edges and $\mathbb{Z}^n$ is associated to a complete graph with n vertices.

We begin by exploring the minimum b_2 problem for arbitrary finitely presented groups. In Section 2 we give a thorough introduction to the invariant, including the calculations for free and free abelian groups. In Section 3 we see how the group cohomology plays an important role in bounding h from below. Specifically, we prove the following useful proposition that holds for finitely presented groups.

Proposition 1.1. *For a finitely presented group G,*

$$2b_2(G) - m_2(G) \leq h(G),$$

where $m_2(G)$ is the maximum rank of the symmetric bilinear form

$$H^2(G;\mathbb{Z}_2) \times H^2(G;\mathbb{Z}_2) \to \mathbb{Z}_2, (a,b) \mapsto (a \cup b) \cap \alpha \tag{1}$$

taken over all choices of $\alpha \in H_4(G;\mathbb{Z}_2)$.

This proposition yields our first theorem for RAAGs:

Theorem 1.2. *If a RAAG G has trivial $H^4(G)$, then $h(G) = 2b_2(G)$.*

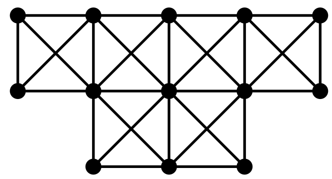
This result holds for all RAAGs with defining graphs of dimension 3 (graphs with no 4-cliques). For graphs of higher dimension, the calculation of h depends on the structure of the graph. Specifically, the geography of 4-cliques (complete subgraphs on 4 vertices) is key to understanding the minimum b_2 problem for the associated RAAG. In dimension 4, the restriction between group theory and topology imposed by Poincaré duality is strengthened through use of the intersection form of a 4-manifold. A portion of the intersection form contains the structure of the 2-dimensional cohomology of the fundamental group, and in Section 4 we see that a RAAG's cohomological structure is completely understood in terms of the configuration of the 4-cliques in the defining graph. We further discuss how one can calculate $m_2(G)$. Most importantly, we suspect that the lower bound given in Proposition 1.1 can always be realized in the case that G is a RAAG, and we prove three inductive theorems which together reduce the minimum b_2 problem to one in which the defining graphs are connected, contain only 4-cliques, and contain no vertices whose removal disconnects the graph.

Theorem 1.3. *Let G_1 and G_2 be RAAGs such that $h(G_i) = 2b_2(G_i) - m_2(G_i)$ for $i = 1, 2$. Then $h(G_1 * G_2) = h(G_1) + h(G_2)$.*

Theorem 1.4. *Let Γ_1 and Γ_2 be two graphs representing RAAGs G_1 and G_2 such that $h(G_i) = 2b_2(G_i) - m_2(G_i)$ for $i = 1, 2$. Let $\{s_1, \ldots, s_m\}$ and $\{t_1, \ldots, t_m\}$ be two sets of pairwise non-adjacent vertices in Γ_1 and Γ_2, respectively. Suppose a new graph, Γ is created by identifying s_i with t_i, $i = 1, \ldots, m$. Then for the RAAG G represented by Γ, $h(G) = h(G_1) + h(G_2)$.*

Theorem 1.5. *Let Γ be a graph associated to a RAAG G. Let r be the number of edges in Γ that are not part of a 4-clique. Suppose the r edges are deleted from Γ resulting in k disjoint subgraphs $\Gamma_1, \ldots, \Gamma_k$. By construction, all the edges in the Γ_i are necessarily part of at least one 4-clique. Let G_i be the group associated to Γ_i. If $h(G_i) = 2b_2(G_i) - m_2(G_i)$ for each i, then $h(G) = \sum_i h(G_i) + 2r$.*

In Section 5 we discuss techniques for constructing manifolds that minimize b_2. These techniques are used in Section 6 to show that for several infinite families of RAAGs, the lower bound in Proposition 1.1 is an equality. For example, in Theorem 6.1 we show how to construct minimal 4-manifolds with RAAG fundamental groups in which the defining graphs consist of 4-cliques with vertices lying on a $\mathbb{Z}^2$ lattice. The following graph is an example.



Other classes of graphs include 4-cliques sharing faces (triangles), and classes with 5-, 6-, and 7-cliques. These results provide evidence for the following conjecture which we present in Section 7:

Conjecture 1.6. *If G is a RAAG, then $h(G) = 2b_2(G) - m_2(G)$.*

Proving this conjecture is not yet accessible, and we conclude by discussing the difficulties in solving this general problem for all RAAGs.

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2. The Hausmann-Weinberger invariant

2.1. **Basic definitions.** In 1985, Hausmann and Weinberger defined the invariant $q(G)$ to be the minimum Euler characteristic over all topological M with fundamental group G. Advances have been made in studying q for classes of groups including knot groups [6], fundamental groups of aspherical manifolds [8,10], free groups, fundamental groups of closed oriented genus g surfaces and 3-manifold groups [10], and most recently finitely generated abelian and free abelian groups [8]. For the cases of infinite amenable groups [5] and groups with finite abelianization [12], L^2-methods have been used to bound q below by zero.

For a 4-manifold M, the Euler characteristic $\chi(M)$ is given by the alternating sum of the ranks of homology (with rational coefficients). These ranks are commonly referred to as Betti numbers; we will denote the ith Betti number by $b_i(M)$. By Poincaré duality, $\chi(M) = 2 - 2b_1(M) + b_2(M)$.

For a group G we can similarly define $b_i(G) = \dim H_i(K(G,1);\mathbb{Q})$, where $K(G,1)$ is an Eilenberg-Maclane space. If G is a finitely presented group with a presentation $\mathscr{P}$ having g generators and r relations, define the *deficiency* $d(\mathscr{P}) = g - r$. Then the *deficiency* d_G of G is the maximum $d(\mathscr{P})$ over all finite presentations $\mathscr{P}$ [6].

A priori, we see that $q(G)$ takes integer values. We have lower and upper bounds on $q(G)$ which allow us to consider q as the minimum rather than the infimum over all $\chi(M)$.

Theorem 2.1 (Hausmann-Weinberger, [6, Theorem 1])**.** *For a finitely presented group G, we have*

$$2 - 2b_1(G) + b_2(G) \leq q(G) \leq 2(1 - d_G).$$

Proof. Let G be a finitely presented group with g generators and r relations such that $d_G = g - r$. Let $M \in \mathcal{M}(G)$ and $f : M \to K(G,1)$ be a map inducing an isomorphism on fundamental groups. The induced map on homology $f_* : H_i(M) \to H_i(G)$ is an isomorphism for $i = 1$ and a surjection for $i = 2$. The surjection in dimension 2 can be seen by considering the Hopf exact sequence, $\pi_2(M) \to H_2(M) \to H_2(\pi_1(M)) \to 0$. Thus $b_1(M) = b_1(G)$ and

$b_2(M) \geq b_2(G)$. To see the upper bound, consider the following construction of a 4-manifold in $\mathcal{M}$: Build a handlebody X consisting of one 0-handle, g 1-handles, and r 2-handles (attached to reflect each of the relations), and double it. The result is a closed orientable 4-manifold M with $\pi_1(M) \cong G$ and $\chi(M) = 2 - 2g + 2r = 2(1 - d_G)$. □

Since $b_1 M = b_1 G$, determining $q(G)$ simplifies to refining the bounds on possible values of $b_2 M$. Kirk and Livingston investigated the q invariant for finitely generated abelian and free abelian groups in [8] and introduced an invariant equivalent to q:

Definition 2.2 (Kirk-Livingston, [8])**.** For a finitely presented group G, define

$$h(G) = \min\{b_2(M) | M \in \mathcal{M}(G)\}.$$

As mentioned in the introduction, we will refer to the problem of determining $h(G)$ for a group G as the *minimum b_2 problem for G*. By definition $q(G) = 2 - 2b_1 G + h(G)$, so solving the minimum b_2 problem for G is equivalent to finding $q(G)$. The following corollary then follows from Theorem 2.1:

Corollary 2.3. *For a finitely presented group G with r relations,*

$$b_2(G) \leq h(G) \leq 2r.$$

The basic technique to solving the minimum b_2 problem is to increase the lower bound on $h(G)$, if possible, and then construct a suitable 4-manifold M with $b_2(M)$ equal to the lower bound, thus yielding an equality. We call such a 4-manifold $M \in \mathcal{M}(G)$ with $b_2(M) = h(G)$ a *realizing manifold for $h(G)$*.

Example. For a free group F_n, $h(F_n) = 0$: Let M be an arbitrary 4-manifold in $\mathcal{M}(F_n)$. We know from Theorem 2.1 that $b_2(F_n) \leq b_2(M)$. A bouquet of n circles is a $K(F_n, 1)$ complex in which $b_2(F_n) = 0$. Thus, $h(F_n)$ is bounded below by zero. One 4-manifold realizing this lower bound is the connected sum of n copies of $S^1 \times S^3$. Since $\pi_1(\#n(S^1 \times S^3)) \cong F_n$ and $b_2(\#n(S^1 \times S^3)) = 0$, $h(F_n) = 0$.

Example. The solution for free abelian groups, a special case of RAAGs, is given in the theorem below:

Theorem 2.4 (Kirk-Livingston, [8, Theorem 1])**.** *For a free abelian group $\mathbb{Z}^n$, $h(\mathbb{Z}^n) = \binom{n}{2} + \epsilon_n$ for all n, with the exception of $h(\mathbb{Z}^3) = 6$ and $h(\mathbb{Z}^5) = 14$. Here ϵ_n is an auxiliary function defined to be 0 if $\binom{n}{2}$ is even and 1 otherwise.*

When $b_2(\mathbb{Z}^n) = \binom{n}{2}$ is odd, the lower bound on h is increased by 1. This argument is explained later by Proposition 4.4. The full details of the proof, including the 4-manifold constructions, can be found in [8].

In the free abelian case, the realizing manifolds are built from products of surfaces that are surgered to identify generators or kill commutators. We shall see that manifolds realizing general RAAGs can be constructed in a similar way.

3. The cohomological obstruction to solving the minimum b_2 problem

3.1. **Finding a better lower bound for h.** Corollary 2.3 asserts that for any finitely presented group G, $b_2(G) \leq h(G)$. We will refer to $b_2(G)$ as the *trivial lower bound* on $h(G)$. In many cases we can use the cohomological structure of G to yield a better lower bound for $h(G)$.

Let $f : M \to K(G,1)$ be a map that induces an isomorphism on fundamental groups, and let $f^* : H^i(G) \to H^i(M)$ be the induced map on cohomology. In the proof of Theorem 2.1 it is shown that the induced homological map $f_* : H_i(M) \to H_i(G)$ is an isomorphism for $i = 1$ and a surjection for $i = 2$. By the Universal Coefficient Theorem, $f^* : H^i(G) \to H^i(M)$ is an isomorphism for $i = 1$ and an injection for $i = 2$. Denote by $I(M,f)$ the image $f^*(H^2(G))$ in $H^2(M)$ modulo torsion.

Consider the symmetric, bilinear pairing

$$H^2(G) \times H^2(G) \to \mathbb{Z} \text{ by } (a,b) \mapsto (a \cup b) \cap \alpha \tag{2}$$

for a homology class $\alpha \in H_4(G)$. If $\alpha = f_*([M])$, this form completely determines the restriction of the intersection form of M,

$$H^2(M)/\text{torsion} \times H^2(M)/\text{torsion} \to \mathbb{Z} \text{ by } (x,y) \mapsto (x \cup y) \cap [M],$$

to $I(M,f)$ since $(f^*(a) \cup f^*(b)) \cap [M] = (a \cup b) \cap \alpha$.

Given any group G and homology class $\alpha \in H_4(G)$, there exists $M \in \mathcal{M}(G)$ and a continuous map $f : M \to K(G,1)$ so that $f_*([M]) = \alpha$ [2]. Additionally, the rank of $I(M,f)$ is $b_2(G)$. These two observations allow us to make certain assumptions about the possible values of $h(G)$ independent of the 4-manifold M or the classifying map $f : M \to K(G,1)$.

We introduce the following definition which is useful for improving the trivial lower bound on $h(G)$ for any finitely presented group G.

Definition 3.1. For a finitely presented group G, define $m(G)$ to be the maximum rank of a matrix associated to (2) over all possible choices of $\alpha \in H_4(G)$.

Note that a priori, $0 \leq m(G) \leq b_2(G)$. If $m(G)$ is strictly less than $b_2(G)$, then $I(M,f)$ is represented by a singular matrix, which indicates the lower bound on $h(G)$ must be greater than $b_2(G)$, the dimension of $I(M,f)$. Unfortunately, computing $m(G)$ is impractical; in all nontrivial cases, there are infinitely many choices of $\alpha \in H_4(G;\mathbb{Z})$. However, $H_4(G;\mathbb{Z}_p)$ can be finite. If p is prime, the intersection form of a 4-manifold M with $\mathbb{Z}_p$ coefficients is also nonsingular. Thus we can calculate $m_p(G)$ instead, a mod p version of $m(G)$.

Definition 3.2. Define $m_p(G)$ to be the maximum rank of the symmetric bilinear form

$$H^2(G;\mathbb{Z}_p) \times H^2(G;\mathbb{Z}_p) \to \mathbb{Z}_p, (a,b) \mapsto (a \cup b) \cap \alpha \tag{3}$$

over all possible choices of $\alpha \in H_4(G;\mathbb{Z}_p)$.

In practice, for RAAGs we need consider only the bilinear form on $H^2(G;\mathbb{Z}_p)$ for $p = 2$; we only use $m_2(G)$, the invariant mentioned in the introduction. We now prove Proposition 1.1 (which holds for all prime p although it is stated in the introduction for $p = 2$).

Proof of Proposition 1.1. Let G be a finitely presented group, and let X be a $K(G,1)$ space. Then $H_1(X)$ and $H_2(X)$ are finitely presented and $b_2(G) = \dim H_2(X;\mathbb{Q}) = \dim H^2(X;\mathbb{Q})$, as we identify $H_*(G)$ with $H_*(X)$ and $H^*(G)$ with $H^*(X)$. Let $\tilde{\alpha}$ be the homology class that maximizes the rank of the form (3) over all $\alpha \in H_4(G; \mathbb{Z}_p)$. Consequently, $\tilde{\alpha}$ minimizes the radical of (3). Recall that for a symmetric bilinear form, the radical contains linear independent vectors x_i such that $\langle x_i \cup y, \alpha \rangle = 0$ for all $y \in H^2(G; \mathbb{Z}_p)$ and a choice of $\alpha \in H_4(G; \mathbb{Z}_p)$. Since the dimension of the form is $b_2(G)$, the minimum dimension of the radical is $b_2(G) - m_p(G)$ by the Rank-Nullity Theorem. In order for the intersection form on a manifold $M \in \mathcal{M}(G)$ to be nondegenerate, its rank must be at least $b_2(G) + (b_2(G) - m_p(G))$. Thus $2b_2(G) - m_p(G) \leq h(G)$. □

4. Right-angled Artin groups

We now restrict our discussion of the minimum b_2 problem to RAAGs. The *Salvetti complex* is a compact $K(G, 1)$ space commonly constructed by attaching higher dimensional tori to a wedge of circles, and is used in the computation of the group cohomology of RAAGs in [3].

Theorem 4.1 (Charney-Davis, [3, Theorem 3.2.4])**.** *Suppose that G is a RAAG with generators $s_1, \ldots, s_n$. Let $\Lambda[y_1, \ldots, y_n]$ be the exterior algebra over $\mathbb{Z}$ on the variables $y_1, \ldots, y_n$. Let I be the ideal generated by all products $y_i y_j$ such that s_i and s_j do not commute in G. Then $H^*(G) \cong \Lambda[y_1, \ldots, y_n]/I$.*

Nontrivial cup products in the cohomology ring of a RAAG come from the commuting generators, as in the case of a torus. This is because the chain complex of a Salvetti complex injects into that of a torus, where all chain (and cochain) maps are trivial.

For a graph Γ associated to a RAAG G, we can recognize generators of $H^*(G)$ straight from the graph Γ: vertices represent generators of $H^1(G)$, edges represent generators of $H^2(G)$, and triangles represent generators of $H^3(G)$. In general, k-cliques, or complete subgraphs of order k, represent generators of $H^k(G)$.

Example. Let Γ be the graph in Figure 1, representing a RAAG, G. We can think of the vertices $\{s_1, \ldots, s_6\}$ as representing generators $\{z_1, \ldots, z_6\}$ of $H^1(G)$. To simplify notation, let $z_i \cup z_j$ be denoted by z_{ij}. Thus z_{12}, z_{13}, z_{14}, z_{15}, z_{23}, z_{24}, z_{25}, z_{34}, z_{35}, z_{36}, z_{45}, z_{46}, and z_{56} represent generators of $H^2(G)$. $H^3(G)$ is generated by z_{123}, z_{124}, z_{125}, z_{134}, z_{135}, z_{145}, z_{234}, z_{235}, z_{245}, z_{345}, z_{346}, z_{356}, and z_{456}, and $H^4(G)$ is generated by z_{1234}, z_{1235}, z_{1245}, z_{1345}, z_{2345}, and z_{3456}. Lastly, z_{12345} generates the top dimensional cohomology class in $H^5(G)$.

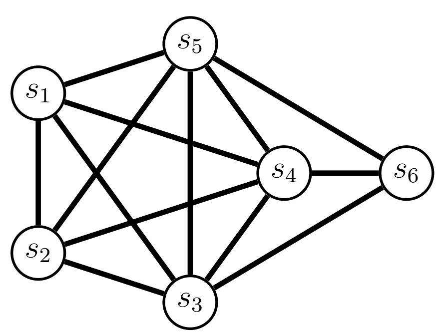


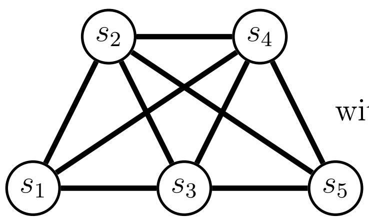
Figure 1. A graph of a 5-clique attached to a 4-clique along a face

With this calculation of the cohomology ring, we can now prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a RAAG with g generators and r relations. Then $b_1(G) = g$ and $b_2(G) = r$. Let M be any 4-manifold in $\mathcal{M}(G)$ and let $f : M \to K(G, 1)$ be a map inducing an isomorphism on fundamental groups. If $H^4(G) = 0$, then the image $I(M, f)$ of any basis of $H^2(G)$ can be represented by a zero matrix of dimension $b_2(G)$. Clearly, $m_2(G) = 0$. Thus $2b_2(G) \leq h(G)$ by Proposition 1.1. By Corollary 2.3, $h(G) \leq 2r = 2b_2(G)$. $\square$

4.1. **Computing $m_2(G)$ for RAAGs.** For a RAAG G, since $H_4(G)$ is finitely generated, a computer program can calculate $m_2(G)$. The following example provides an algorithm for computing $m_2(G)$ given an adjacency matrix of its defining graph.[1]

Example 4.2. Let G be the group given by the following graph Γ



with adjacency matrix $\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$.

The set of vertices $\{s_1, \ldots, s_5\}$ give an ordered basis for $H_1(G)$. Consider the following matrix representing the form (1) under the ordered basis $\{s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{25}, s_{34}, s_{35}, s_{45}\}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & a_2 \\ 0 & a_1 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero elements of this matrix are variables a_1 and a_2 representing the two generators s_{1234} and s_{2345} of $H^4(G; \mathbb{Z}_2)$. We compute $m_2(G)$ by finding all $2^{b_4(G)}$ ranks of the form and taking the maximum. Each rank is computed by replacing the a_i in the above matrix with ones or zeros, each entry representing $\langle a_i, \alpha \rangle$. Consequently, we find $m_2(G) = 6$. This implies that the minimum dimension of the radical is $b_2(G) - m_2(G) = 9 - 6 = 3$. However, since $b_4(G)$ is not too large, we can compute $m_2(G)$ easily by hand by calculating the minimum dimension of the radical.

There are three nonzero choices in $H_4(G; \mathbb{Z}_2)$ for α: α_1, α_2, and $\alpha_1 + \alpha_2$, where $\langle a_i, \alpha_j \rangle = \delta_{ij}$. Note that if $\alpha = 0$, the rank of the matrix is zero and the nullity is $b_2(G) = 9$. If $\alpha = \alpha_1$, then $\langle a_1, \alpha \rangle = 1$ and $\langle a_2, \alpha \rangle = 0$. In replacing a_1 with 1 and a_2 with 0, we see that this matrix has nullity 3. Similarly, if $\alpha = \alpha_2$, then we replace a_1 with 0 and a_2 with 1 and the

[1]One such program has been implemented by the author in SAGE and can be located in Appendix A of the preprint posted on the arXiv: http://arxiv.org/abs/1401.2478.

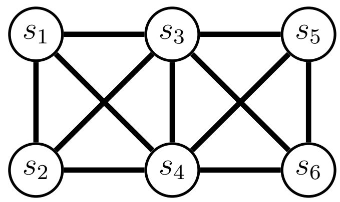
matrix again has nullity 3. If $\alpha = \alpha_1 + \alpha_2$, then we replace both a_1 and a_2 with 1. Three rows of the matrix (namely the fourth, fifth, and seventh) have two nonzero elements. The three linearly independent vectors

$$[0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1]^t,\ [0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0]^t,\ [1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0]^t$$

are in the kernel of the matrix over $\mathbb{Z}_2$. Since the matrix represents the form (1), the dimension of the radical for $\alpha = \alpha_1 + \alpha_2$ is 3. Thus the minimum dimension of the radical is 3. Equivalently, the maximum rank is 6.

In some cases, it is not difficult to calculate $m_2(G)$ by determining the minimum dimension of the radical of $H^2(G, \mathbb{Z}_2)$ directly from the defining graph of G. In the following example, we let $\{s_i\}$ be a basis for the homology and $\{z_i\}$ be the dual basis for the cohomology.

Example 4.3. Let the graph of G be given below:



This graph is made up of exactly two 4-cliques, so $b_4(G) = 2$. Since each 4-clique has 6 edges, we can compute $b_2(G)$ by multiplying 6 by the number of 4-cliques and subtracting the number of *shared edges*, edges that belong to more than one 4-clique. In this example, $b_2(G) = 6(2) - 1 = 11$. Define the two generators s_{1234} and s_{3456} of $H_4(G)$ to be α_1 and α_2, respectively. Let α be an arbitrary element of $H_4(G; \mathbb{Z}_2)$. Then α is of the form $c_1\alpha_1 + c_2\alpha_2$, where c_1 and c_2 are either 0 or 1. There are only three nontrivial choices for α. If $c_1 = 0$, then z_{13}, z_{14}, z_{23}, and z_{24} give a basis for the radical of the form (1). For any generator z of $H^2(G)$, $\langle z_{13} \cup z, \alpha_2\rangle = 0$, $\langle z_{14} \cup z, \alpha_2\rangle = 0$, $\langle z_{23} \cup z, \alpha_2\rangle = 0$, and $\langle z_{24} \cup z, \alpha_2\rangle = 0$. This is shown in the graph since each of the corresponding four edges (s_{13}, s_{14}, s_{23}, and s_{24}) are only part of the 4-clique α_1. Similarly, if $c_2 = 0$, then z_{35}, z_{36}, z_{45}, and z_{46} give a basis for the radical.

Lastly, consider the case when both $c_1 = 1$ and $c_2 = 1$. Consider the image of $z_{12} + z_{56}$ cupped with an arbitrary generator z under the form (1):

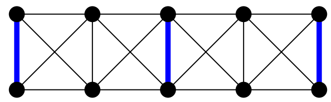
$$\langle (z_{12} + z_{56}) \cup z, \alpha_1 + \alpha_2\rangle = \langle z_{12} \cup z, \alpha_1\rangle + \langle z_{56} \cup z, \alpha_1\rangle + \langle z_{12} \cup z, \alpha_2\rangle + \langle z_{56} \cup z, \alpha_2\rangle$$

On the right-hand side, the middle two summands are zero, and $\langle z_{12} \cup z, \alpha_1\rangle$ and $\langle z_{56} \cup z, \alpha_2\rangle$ are zero unless $z = z_{34}$. If $z = z_{34}$, then $\langle z_{12} \cup z_{34}, \alpha_1\rangle + \langle z_{56} \cup z_{34}, \alpha_2\rangle = 1 + 1 \equiv 0 \mod 2$. One can check other linearly independent elements of $H^2(G; \mathbb{Z}_2)$ and see that this unique element provides a basis for the radical. Thus the maximum rank of the form is 10 instead of 11. This gives the lower bound $12 \leq h(G)$.

Consider next a graph of three 4-cliques attached edge-to-edge, and an arbitrary element $\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 \in H_4(G, \mathbb{Z}_2)$, $c_i \in \{0, 1\}$. If any $c_i = 0$, the nullity of the form is at least 4, for the same reason as in the above case. Thus we may assume $\alpha = \alpha_1 + \alpha_2 + \alpha_3$. One can verify that there are no nonzero elements in the radical.

In a graph of four 4-cliques attached edge-to-edge, we have the same assumption that the minimum nullity of the form occurs with the choice $\alpha = \alpha_1 + \ldots + \alpha_4$. Again, the nullity is

1; the element in this radical is the sum of the generators represented by the **bold** edges in the following graph:



A pattern develops that indicates that in graphs with a string of k 4-cliques attached edge-to-edge, the nullity is either 0 or 1, depending on the parity of k.

For many RAAGs including some of those in Example 4.3, the following proposition allows us to increase the trivial lower bound on $h(G)$ by 1 in the case when $b_2(G)$ is odd.

Proposition 4.4. *If G is a RAAG, $m_2(G)$ is even. Thus if $b_2(G)$ is odd, $b_2(G)+1 \leq h(G)$.*

Proof. Let $\{z_i\}$ be the set of generators of $H^1(G)$. Any nonzero generator of $H^2(G)$ is of the form $z_i \cup z_j$. Under the cup product map in (1), $\langle (z_i \cup z_j)^2, \alpha\rangle$ is zero for any choice of α, since the z_i are odd dimensional homology classes. A bilinear form $B : V \times V \to GF(q)$ is considered *alternating* if $B(x, x) = 0$ for all $x \in V$. Thus (1) is an alternating form. In [4, Lemma 10] it is shown that alternating bilinear forms over $GF(q)$ have even rank. For $GF(q) = \mathbb{Z}_2$, we see that the rank must be even, and thus $m_2(G)$ must be even. If $b_2(G)$ is odd, then $m_2(G)$ is at most $b_2(G) - 1$ by Proposition 1.1, and so $b_2(G) + 1 \leq h(G)$. ☐

Alternatively, one can bound $h(G)$ from below by finding a maximum isotropic subspace of the form (1). This yields the same calculation of the lower bound from Proposition 1.1, since twice the dimension of a maximum isotropic subspace of $H^2(G;\mathbb{Z}_2)$ is equal to $2b_2(G) - m_2(G)$. In some cases we can find a subset of the generators of $H^2(G;\mathbb{Z}_2)$ that form a maximum isotropic subspace, which are represented in the defining graph as edges.

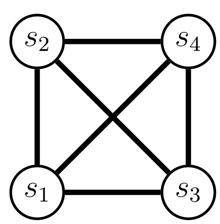


FIGURE 2. A 4-clique, the defining graph for $\mathbb{Z}^4$.

Consider the graph of a 4-clique in Figure 2. The vertices $\{s_i\}$ determine an ordered basis $\{z_i\}$ for $H^1(\mathbb{Z}^4)$. Then $\{z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34}\}$ represent edges of the graph, and z_{1234} represents the 4-clique. The following two sets give maximum isotropic subspaces for $H^2(\mathbb{Z}^4)$: $\{z_{12}, z_{24}, z_{14}\}$ and $\{z_{12}, z_{24}, z_{23}\}$. In each set, every pair of generators is either of the form (z_{ij}, z_{jk}), (z_{ij}, z_{ik}), or (z_{ij}, z_{jk}). In every pair, the product of the two generators is zero because $z_{ii} = 0$ for all i. The edges represented by the two sets above form a triangle and a claw, respectively. The two isotropic subspaces are **highlighted** in Figure 3 (a). Of course, these sets are not the only choices for maximum isotropic subspaces for a 4-clique. However, any three dimensional isotropic subspace of a 4-clique will either form a triangle or a claw in the graph.

Consequently, pairs of generators (z_{ij}, z_{kl}) will cup nontrivially if i, j, k, l are all distinct. Therefore in every 4-clique of a graph, the maximum isotropic subspace will never contain any pair of bold edges shown in Figure 3 (b).

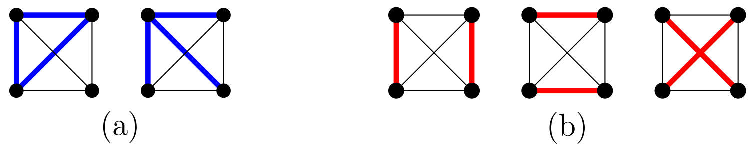


Figure 3. Triangles and claws, formed by the **bold** edges in (a), are subgraphs that make up an isotropic subspace in each 4-clique. Pairs of edges **highlighted** in (b) do not.

4.2. **Cohomologically minimal groups.** The main question we will discuss in this paper is the following: *For a RAAG G, is $h(G)$ determined entirely by the structure of $H^*(G)$?*

Let us make the following definition.

Definition 4.5. We say that a finitely presented group G is *cohomologically minimal* if $h(G) = 2b_2(G) - m_2(G)$.

Restricting our discussion of the minimum b_2 problem to cohomologically minimal RAAGs, we now prove Theorems 1.3, 1.4, and 1.5.

Proof of Theorem 1.3. By assumption, $h(G_1) = 2b_2(G_1) - m_2(G_1)$ and $h(G_2) = 2b_2(G_2) - m_2(G_2)$. For the free product $G_1 * G_2$, $m_2(G_1 * G_2) = m_2(G_1) + m_2(G_2)$ and $b_2(G_1 * G_2) = b_2(G_1) + b_2(G_2)$. To see the former statement, note that the bilinear form under the free product splits into a direct sum of forms. For the latter statement, note that homology is additive under free products. This gives a lower bound on $h(G_1 * G_2)$:

$$\begin{aligned} 2(b_2(G_1) + b_2(G_2)) - (m_2(G_1) + m_2(G_2)) &\leq h(G_1 * G_2) \\ h(G_1) + h(G_2) &\leq h(G_1 * G_2) \end{aligned}$$

Let M_i be a realizing manifold for $h(G_i)$; that is, $\pi_1(M_i) \cong G_i$ and $b_2(M_i) = h(G_i)$. Then $b_2(M_1 \# M_2) = b_2(M_1) + b_2(M_2) = h(G_1) + h(G_2)$. Therefore $h(G_1 * G_2) = h(G_1) + h(G_2)$. Note that this implies one realizing manifold for $h(G_1 * G_2)$ is the connected sum of the realizing manifolds for $h(G_1)$ and $h(G_2)$. $\square$

Proof of Theorem 1.4. The proof of this theorem is very similar to that of Theorem 1.3. Let G_i be the RAAG associated to Γ_i. By assumption, $h(G_1) + h(G_2) = 2(b_2(G_1) + b_2(G_2)) - (m_2(G_1) + m_2(G_2))$. We will first see that by identifying generators of G_1 and G_2, we do not create any new 4-cliques; this will assert that $m_2(G) = m_2(G_1) + m_2(G_2)$.

Suppose that by identifying s_i with t_i and s_j with t_j we create a 4-clique involving the two newly identified generators. This would require an edge between either s_i and s_j or t_i and t_j. However, we have assumed both the $\{s_i\}$ and $\{t_i\}$ are pairwise non-adjacent. No edges in Γ_1 will form a 4-clique with edges in Γ_2 after the identifications of the vertices, so the bilinear form for $H^2(G)$ splits into a direct sum of forms for $H^2(G_1)$ and $H^2(G_2)$. Thus $h(G_1) + h(G_2) \leq h(G)$.

Let M_i be a realizing manifold for $h(G_i)$. We can build a realizing manifold M for $h(G)$ by taking $M_1 \# M_2$ and performing m surgeries, each identifying s_i with t_i. These surgeries do not alter the second homology of the manifold, as we will see in Section 5.1. Thus, M has $b_2(M) = b_2(M_1) + b_2(M_2) = h(G_1) + h(G_2)$. $\square$

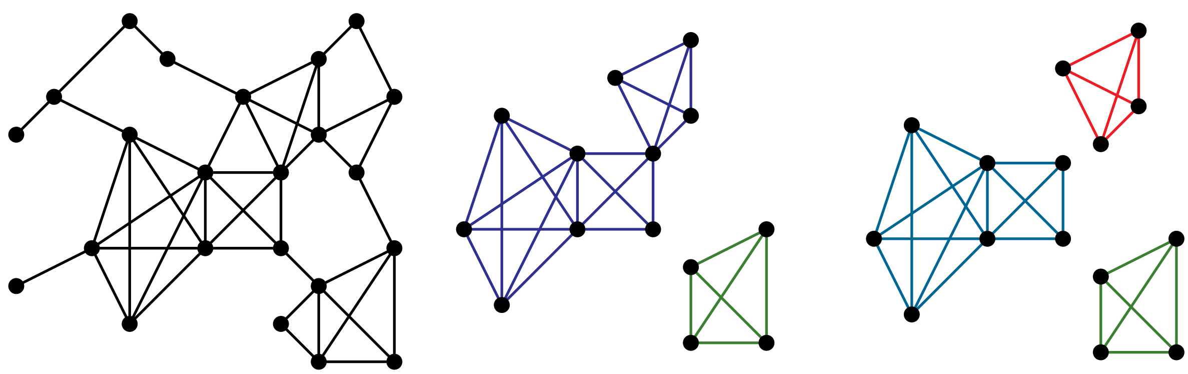
Proof of Theorem 1.5. The r edges deleted from Γ represent basis elements of $H^2(G)$ (as do all edges of Γ) and necessarily cup to zero with any other basis element under (1), and so they are in the radical. This and the assumption that each G_i is cohomologically minimal imply that $m_2(G) = \sum_i m_2(G_i)$. Note also that $b_2(G) = \sum_i b_2(G_i) + r$. Therefore we have the following lower bound on $h(G)$:

$$2(b_2(G_1) + \cdots + b_2(G_k) + r) - (m_2(G_1) + \cdots + m_2(G_k)) = h(G_1) + \cdots h(G_k) + 2r \leq h(G).$$

Let M_i be a realizing manifold for $h(G_i)$. Build a realizing manifold M for $h(G)$ by starting with the connected sum $M_1 \# \cdots \# M_k$ and performing r surgeries to introduce the relations we initially ignored from G. Each surgery increases b_2 by 2, as we will see in Section 5.1. These surgeries yield a 4-manifold M with $\pi_1(M) = G$ and $b_2(M) = \sum_i b_2(M_i) + 2r = \sum_i h(G_i) + 2r$. $\square$

These theorems reduce the complexity of the minimum b_2 problem for RAAGs; we need only consider the case where the defining graph Γ is connected, with all edges belong to at least one 4-clique, and has no *cut vertices*. In graph theory, a *cut vertex* is any vertex whose removal disconnects the graph. (Graphs with cut vertices need not be considered by Theorem 1.4.)

Example 4.6. Let G be a RAAG with defining graph Γ in Figure 4 (a). Using the above theorems, we can break down the calculation of $h(G)$ into calculations for three different groups.



(a) (b) (c)

FIGURE 4. An example of the breakdown of a graph into disjoint subgraphs, for the calculation of h according to Theorems 1.4 and 1.5

By removing 16 edges in Γ that are not part of a 4-clique, we get two disjoint graphs in Figure 4 (b). Call these two graphs Γ_1 and Γ_2. Assuming the resulting RAAGs G_1

and G_2 associated to Γ_1 and Γ_2 are cohomologically minimal, Theorem 1.5 asserts that $h(G) = h(G_1) + h(G_2) + 2(16)$. Furthermore, the graph on the left in Figure 4 (b) has two subgraphs joined at one vertex. By splitting the subgraphs apart, we have the three disjoint graphs in Figure 4 (c). Call these disjoint graphs Γ_a, Γ_b, and Γ_c. Theorem 1.4 asserts that $h(G_1)+h(G_2) = h(G_a)+h(G_b)+h(G_c)$, under the assumption that G_a, G_b, and G_c are each cohomologically minimal. Together, we have

$$h(G) = h(G_a) + h(G_b) + h(G_c) + 32.$$

Indeed, the groups corresponding to the graphs in Figure 4 (c) are cohomologically minimal. In Section 5.2 we will complete the calculation of $h(G)$ by calculating $h(G_a)$, $h(G_b)$, and $h(G_c)$. See Example 5.4 for details.

5. Tools for 4-manifold constructions

In this section, we describe surgery methods used to build realizing 4-manifolds with a given RAAG fundamental group. Lemma 5.1 and Theorem 5.3 describe surgery methods that increase b_2 of a 4-manifold without changing its fundamental group and Lemma 5.5 describes a situation in which it is possible to reduce b_2 via surgery on dual spheres. Lemma 5.7 describes a specific space in which multiple sphere surgeries can be performed, and is a basic building block for a family of examples provided in Section 6. This section also provides a helpful way of viewing the effects of these 4-manifold surgeries on the graphs of their fundamental groups.

5.1. **Tools from [8].** We will make use of the following classical result.

Lemma 5.1 (Milnor, [13, Lemma 2])**.** *If a 4-manifold M' is constructed from a compact 4-manifold M via surgery along a curve γ, then $b_2(M') = b_2(M)$ if γ is of infinite order in $H_1(M)$ and $b_2(M') = b_2(M) + 2$ otherwise.*

Proof. Surgery on M is performed by removing $S^1 \times B^3$ and replacing it with $D^2 \times S^2$, so $\chi(M') = \chi(M) + 2$. If γ is of infinite order in $H_1(M)$, $b_1(M') = b_1(M) - 1$ and $b_3(M') = b_3(M) - 1$. Thus the difference in Euler characteristic comes from the change in b_1 and b_3, so $b_2(M') = b_2(M)$. If γ is of finite order, b_1 and b_3 are unchanged, so the difference in Euler characteristic comes from an increase in b_2 by 2. □

We will use this lemma to perform two types of surgeries on curves in a 4-manifold. The first type is a surgery that identifies generators a and b of the fundamental group; the surgery curve is $\gamma = ab^{-1}$, which has infinite order in H_1. The second type is a surgery that creates a commutator relation. Performing surgery on the nullhomologous curve $\gamma = aba^{-1}b^{-1}$ kills the commutator of a and b. Lemma 5.1 implies that performing surgery to identify generators does not change b_2, whereas a surgery to kill a commutator increases b_2.

The next definition and subsequent theorem were developed in [8] and are extremely useful in constructing realizing manifolds for RAAGs.

Definition 5.2 (Kirk-Livingston, [8, Definition 5])**.** A *4-reduction of a group G by a 4-tuple of elements* $[w_1, w_2, w_3, w_4]$, $w_i \in G$, is the quotient of G by the normal subgroup generated by the 6 commutators $[w_i, w_j]$, $i < j$. This quotient is denoted $G/[w_1, w_2, w_3, w_4]$.

More generally, we say a group G can be *4-reduced to the group H using the 4-tuples* $\{[w_{1k}, w_{2k}, w_{3k}, w_{4k}]\}$, $k = 1, \dots, \ell$ if H is isomorphic to the quotient of G by the normal subgroup generated by the 6ℓ commutators $[w_{ik}, w_{jk}]$, $i < j$, $k = 1, \dots, \ell$.

Theorem 5.3 (Kirk-Livingston, [8, Theorem 6])**.** *If M is a 4-manifold and $w_i \in \pi_1(M)$ for $i = 1, \dots, 4$, then there is a 4-manifold M' with $\pi_1(M') = \pi_1(M)/[w_1, w_2, w_3, w_4]$ and $b_2(M') = b_2(M) + 6$.*

Proof. Form the connected sum $M\#T^4$ which increases b_2 by 6. Let $\pi_1(T^4)$ be generated by $\{x_1, x_2, x_3, x_4\}$. Perform surgery on 4 curves $x_i w_i^{-1}$, $i = 1, \dots, 4$ to identify the generators of $\pi_1(T^4)$ with the elements w_i. By Lemma 5.1, these surgeries do not change b_2 since they are of infinite order in $H_1(M\#T^4)$. The effect of the surgeries is that each of the elements w_i commute with each other, so M' is a manifold with the fundamental group claimed. □

5.2. **Graphical representations of fundamental groups.** Many realizing 4-manifolds contain connected sums of 4-tori and other products of surfaces. It is very convenient to view 4-manifolds from the perspective of the graphs of their fundamental groups, if possible.

First let us consider the product of a torus T^2 with a genus 2 surface Σ_2, with π_1 generated by $\{x_1, x_2\}$ and $\{y_1, y_2, y_3, y_4\}$. This 4-manifold has a commutator relation between x_1 and x_2 as well as commutator relations between the x_i and y_j. These we can represent in a graph of the fundamental group as edges between the corresponding vertices. In addition to the commutator relations we have the surface relation $[y_1, y_2][y_3, y_4] = 1$, so this 4-manifold does not have a RAAG as its fundamental group. However, for convenience, let us display the surface relation as two dashed edges, one between y_1 and y_2 and the other between y_3 and y_4, as in Figure 5.

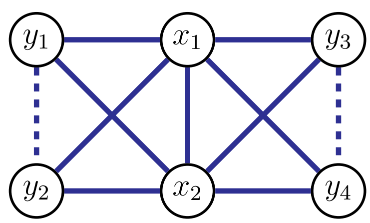


FIGURE 5. A graph representing $\pi_1(T^2 \times \Sigma_2)$

Note that if we perform surgery to either introduce the commutator relation $[y_1, y_2] = 1$ or $[y_3, y_4] = 1$, or the relation is introduced another way (for example, by a 4-reduction), then the resulting 4-manifold has a RAAG as its fundamental group.

Example 5.4. Return to the graph Γ from Example 4.6. Two of the disjoint subgraphs in Figure 4 (c) are 4-cliques. Without loss of generality, let these be Γ_a and Γ_b. Both groups are copies of $\mathbb{Z}^4$, and $h(\mathbb{Z}^4) = 6$. The third graph, Γ_c, consists of a 5-clique and a 4-clique sharing one edge. By calculating $m_2(G_c) = 12$ and $b_2(G_c) = 15$, we know $18 \leq h(G_c)$. A realizing manifold for $h(G_c)$ is built as follows: Start with $(T^2 \times \Sigma_2)\#T^4$, with π_1 generated by $\{x_1, x_2\}$, $\{y_1, y_2, y_3, y_4\}$, and $\{z_1, z_2, z_3, z_4\}$ and $b_2 = 16$. Perform surgery to identify y_1 with z_2, x_1 with z_3, and z_4 with y_2. These surgeries do not change b_2. Finally perform surgery to introduce the commutator relation $[z_1, x_2] = 1$. This surgery increases b_2 by 2 and yields a manifold $M \in \mathcal{M}(G_c)$ with $b_2(M) = 18$. Thus $h(G_a) = 6$, $h(G_b) = 6$, and $h(G_c) = 18$, and all groups are cohomologically minimal.

Recall from Example 4.6 that 16 edges were deleted from Γ. By Theorems 1.4 and 1.5, we know $h(G) = h(G_a) + h(G_b) + h(G_c) + 32$. Thus $h(G) = 6 + 6 + 18 + 32 = 62$.

5.3. **4-reductions in action.** Many realizing manifolds are constructed with 4-reductions, so it is helpful to see these reflected in the graphs of their fundamental groups. Let us begin with a 4-manifold $M = \#5(S^1 \times S^3)$ in which $b_2(M) = 0$. Let the generators of $\pi_1(M)$ be $\{x_1, \ldots, x_5\}$ as shown in Figure 6 (a). Perform a 4-reduction on $[x_1x_5, x_2, x_3, x_4]$ to construct a 4-manifold M'. Recall that each 4-reduction consists of taking a 4-torus and identifying its generators with those in the 4-reduction.

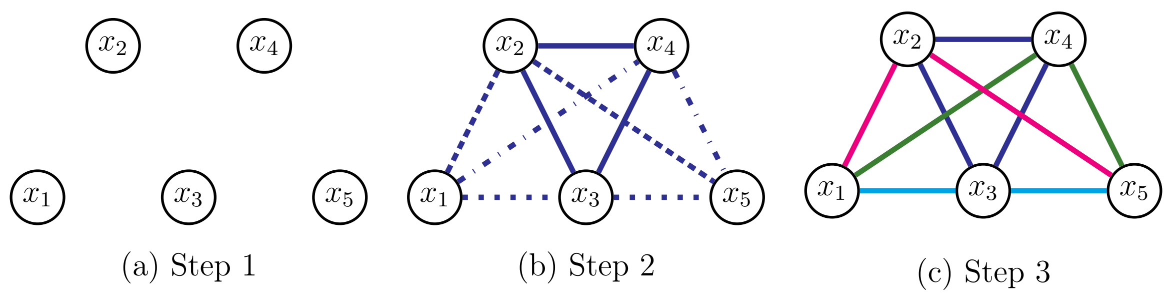


FIGURE 6. A graph showing the path of edges created by the 4-reduction $[x_1x_5, x_2, x_3, x_4]$

Let us look a representation of the graph of $\pi_1(M')$ in Figure 6 (b). The solid lines indicate the existence of the commutator relations between x_2, x_3, and x_4 given by the 4-reduction. The three remaining relations created by the 4-reduction are $[x_1x_5, x_2] = 1$, $[x_1x_5, x_3] = 1$, and $[x_1x_5, x_4] = 1$. We will refer to these types of commutator relations as *surface-like relations* and we can view them as products of commutators. We consider $[x_1x_5, x_2] = 1$ and $[x_1, x_2][x_5, x_2] = 1$ equivalent relations since they represent the same commutator information. More formally, they normally generate the same subgroup. In the same way, we consider $[x_1x_5, x_3] = 1$ equivalent to $[x_1, x_3][x_5, x_3] = 1$ and $[x_1x_5, x_4] = 1$ equivalent to $[x_1, x_4][x_5, x_4] = 1$.

Graphically, we will represent surface-like relations by dashed or dotted lines, as we did in Section 5.2 with the surface relation of $\pi_1(\Sigma_2)$. Since we have three such relations, we can resemble them by three different styles of lines in the graph: dashed, dotted, and a combination of dashes and dots.

Now perform surgery to introduce the following relations: $[x_1, x_2] = 1$, $[x_1, x_3] = 1$, and $[x_1, x_4] = 1$. Because of the surface-like relations introduced by the 4-reduction, we get three relations for free: $[x_2, x_5] = 1$, $[x_3, x_5] = 1$, and $[x_4, x_5] = 1$. The resulting π_1 graph is in Figure 6 (c).

Consider a similar 4-reduction beginning with a manifold $M = \#6(S^1 \times S^3)$, with fundamental group generated by $\{x_1, \ldots, x_6\}$, as shown in Figure 7 (a). Perform the following 4-reduction: $[x_1x_3x_6, x_2, x_4, x_5]$. In Figure 7 (b), the solid lines represent the three commutator relations between x_2, x_4, and x_5. The remaining three relations from the 4-reduction can be represented by the surface-like relations

$$[x_1, x_2][x_2, x_3][x_2, x_6] = 1$$
$$[x_1, x_4][x_3, x_4][x_4, x_6] = 1$$
$$[x_1, x_5][x_3, x_5][x_5, x_6] = 1$$

and are demonstrated by dashed, dotted, and dash-dotted lines. The following commutator surgeries result in a 4-manifold with the π_1 graph in Figure 7 (c):

$$[x_1, x_2], [x_2, x_3], [x_1, x_4], [x_4, x_6], [x_3, x_5], [x_5, x_6].$$

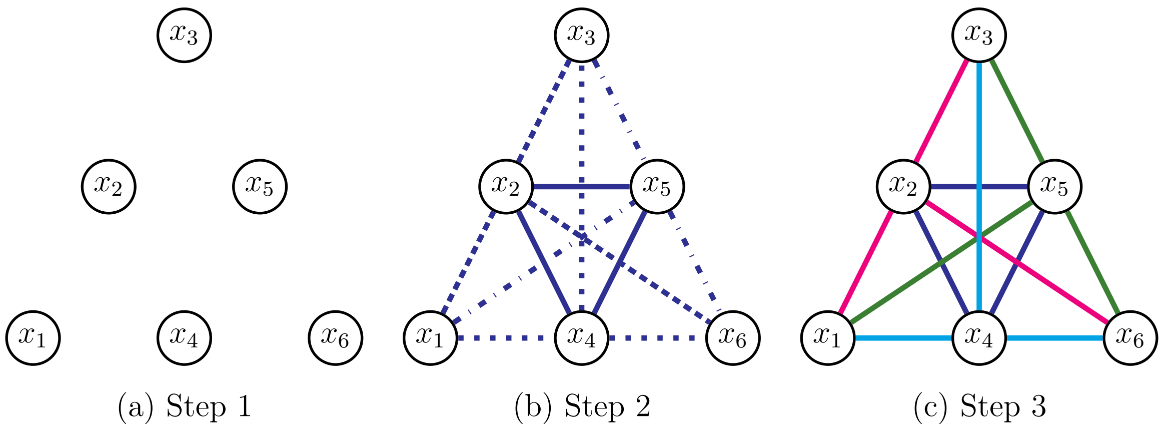
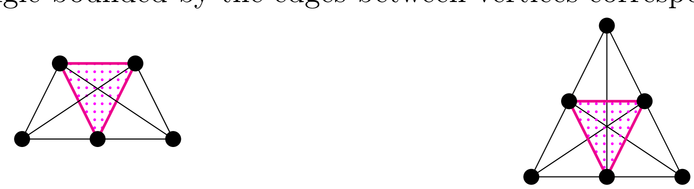


FIGURE 7. A graph showing the path of edges created by the 4-reduction $[x_1x_3x_6, x_2, x_4, x_5]$

These are two examples of 4-reductions of the form $[a, b, c, de \ldots]$, where a, b, and c are generators of π_1 and the fourth element is a product of generators. If many of these types of 4-reductions are required in the construction of a realizing 4-manifold, it is useful to highlight the three commutator relations between a, b, and c. In the two graphs below, we can **shade** the area of the triangle bounded by the edges between vertices corresponding to a, b, and c.



This triangle represents the face that is shared by all 4-cliques whose fourth vertex is represented in the product of the last element of the 4-reduction. This shading technique will be useful in Section 6.2 when we consider graphs of many 4-cliques attached along triangles.

Note that 4-reductions are not limited to the form $[a, b, c, de \ldots]$ above. Each entry may involve many products of generators. The two examples given in this section are included to illustrate the use of 4-reductions for graphs of certain RAAGs we discuss in Section 6.2.

5.4. **Surgery on dual spheres.** Consider the following construction of the connected sum of three 4-tori, with π_1 generated by $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4\}$, and $\{z_1, z_2, z_3, z_4\}$, as shown in Figure 8.

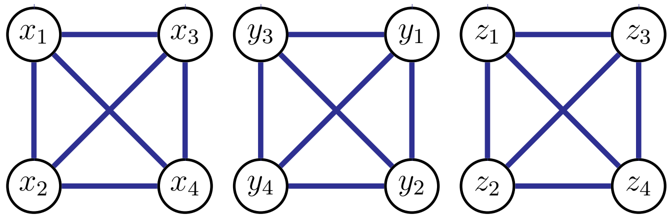


FIGURE 8. A graph representing $\pi_1(T^4\#T^4\#T^4)$

After surgery to identify the generators x_3 with y_3 as well as x_4 with y_4, we can find an embedded 2-sphere in the resulting 4-manifold. View the first 4-torus as the product of two 2-tori, $x_1 \times x_2$ and $x_3 \times x_4$, and the second 4-torus as the product of $y_1 \times y_2$ and $y_3 \times y_4$. We can view the connected sum ambiently and after the identification surgeries, we see a 2-sphere embedded in the 4-manifold.

Similarly, identifying y_1 with z_1 and y_2 with z_2 via surgery creates a second embedded 2-sphere. Because we can initially view the middle 4-torus as the product of two 2-tori which intersect in exactly 1 point, so do the two embedded 2-spheres. We will refer to such a pair of embedded 2-spheres intersecting in this way as a pair of *dual 2-spheres.*

We have already seen in Lemma 5.1 that performing surgery to identify generators of π_1 does not change b_2, and performing surgery to introduce a commutator relation increases b_2 by two. By the next lemma, we can surger out a pair of dual 2-spheres without changing the fundamental group and also *decrease* b_2 by two.

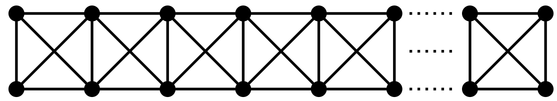
Lemma 5.5. *Suppose in 4-manifold M there exist two 2-spheres intersecting exactly once with at least one embedded with trivial normal bundle. Then it is possible to remove both spheres via surgery without changing the fundamental group of M and also decrease $b_2(M)$ by 2.*

Proof. Suppose S is an embedded 2-sphere in a 4-manifold M, with self-intersection zero. Let $M' = M - S \times B^2$. Then M is built from M' by adding a 2-handle to a nullhomotopic curve and then adding a 4-handle. Neither handle addition changes π_1. Let M_S be the resulting manifold after surgery on S. M_S is built from M' by adding a 3-handle and a 4-handle, thus π_1 remains unchanged. The homology classes of both S and the second 2-sphere are killed by the surgery, thus the rank of $H_2(M;\mathbb{Q})$ decreases by two. □

Remark. Note that this lemma gives a slightly stronger result than what we need, since it allows for one sphere to be immersed. In practice, however, we will always refer to this lemma when surgering out a pair of embedded dual 2-spheres.

This is the only technique we will use to decrease b_2 in certain 4-manifolds. Moreover, for 4-manifolds with π_1 graphs of 4-cliques with more than one pair of dual spheres, in many cases we can surger many if not all pairs of embedded dual spheres to minimize b_2.

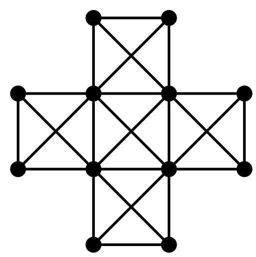
Example 5.6. Consider the setup of the following row of k 4-cliques attached edge to edge, as in the graph below:



Just as before, the way to construct a 4-manifold with minimum b_2 is to start with the connected sum of k 4-tori, and perform surgery to identify the appropriate generators of π_1. Each pair of surgeries identifying the generators of one 4-tori with another creates an embedded 2-sphere, and each sphere intersects one before it and one after it (except the first and last sphere, respectively, where they intersect a 2-torus each). Thus for k 4-cliques as shown above, we have a chain of $k-1$ 2-spheres, with a 2-torus on each end. We can make $\lfloor \frac{k-1}{2} \rfloor$ pairs of dual spheres disjoint by handle slides, and thus perform $\lfloor \frac{k-1}{2} \rfloor$ surgeries on these dual sphere pairs to decrease b_2.

The following lemma will be useful in the following section when we calculate h for certain examples of RAAGs.

Lemma 5.7. *Suppose we have the following graph representing a RAAG G:*



Consider the 4-manifold, M, constructed by the connected sum of five 4-tori with identification surgeries so that the fundamental group has the defining graph above. Then there are two pairs of dual 2-spheres in M. Moreover, we can perform the identification surgeries in such away to make the dual sphere pairs disjoint from each other and thus perform surgery on each pair, thereby decreasing $b_2(M)$ by four.

Proof. As stated in the lemma, begin with the connected sum of five 4-tori, each generated by $\{x_1, x_2, x_3, x_4\}$, $\{a_1, a_2, a_3, a_4\}$, $\{b_1, b_2, b_3, b_4\}$, $\{c_1, c_2, c_3, c_4\}$, and $\{d_1, d_2, d_3, d_4\}$. Denote by X the "middle" 4-torus generated by the x_i, and view X as $[0,1]^4/(0 \sim 1)$. Consider the following identifications of four submanifolds of X, where x_i' is a push-off of x_i:

$$\begin{aligned}
x_1 \times x_2 &= [0,1] \times [0,1] \times \{0\} \times \{0\}/\sim \\
x_3 \times x_4 &= \{0\} \times \{0\} \times [0,1] \times [0,1]/\sim \\
x_1' \times x_3' &= [0,1] \times \{\tfrac{1}{2}\} \times [0,1] \times \{\tfrac{1}{2}\}/\sim \\
x_2' \times x_4' &= \{\tfrac{1}{2}\} \times [0,1] \times \{\tfrac{1}{2}\} \times [0,1]/\sim .
\end{aligned}$$

We can see that $(x_1 \times x_2) \cap (x_3 \times x_4) = (0,0,0,0)$ and $(x_1' \times x_3') \cap (x_2' \times x_4') = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, but the intersections between the other pairs are empty. Thus we can perform the following identifications via surgery: $b_1 = x_1$, $b_2 = x_2$, $d_3 = x_3$, $d_4 = x_4$, $a_1 = x_1'$, $a_3 = x_3'$, $c_2 = x_2'$, and $c_4 = x_4'$. After the identification surgeries, we get two distinct strings of three 4-cliques, representing the existence of two disjoint pairs of dual spheres. We can perform surgery on both of these dual sphere pairs, and by Lemma 5.5, decrease b_2 by four. $\square$

6. Examples of cohomologically minimal RAAGs

We have already seen that a RAAG of dimension 3 is cohomologically minimal. In the first three examples of this section we focus on RAAGs of dimension 4. We begin in Section 6.1 with graphs consisting of multiple 4-cliques attached along edges, and continue in Section 6.2 with attachments along faces (triangles). We end in Section 6.3 with examples of graphs containing 5-, 6-, and 7-cliques.

Due to Theorems 1.3, 1.4 and 1.5, all graphs are connected, have no cut vertices, and all edges belong to at least one 4-clique. Also, it should be noted that any reference to the *radical* or *(maximum) isotropic subspace* is in reference to those of the form (1).

6.1. **Grids of 4-cliques sharing edges.** Consider the family of RAAGs that have defining graphs composed of 4-cliques attached edge-to-edge in a grid pattern, aligned in rows and columns so that the vertices lie on a $\mathbb{Z}^2$ lattice. Figure 9 shows some examples. We refer to these graphs as members of the *Grid family.*

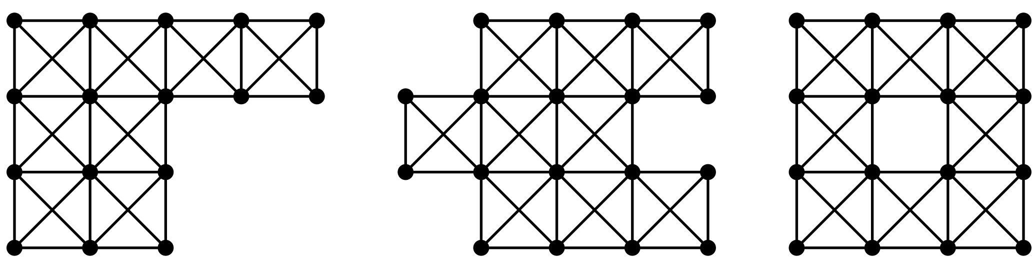


Figure 9. Examples of graphs in the Grid Family

Theorem 6.1. *Let G be a RAAG with a defining graph belonging to the Grid family. Then G is cohomologically minimal.*

Proof. Let Γ be the graph associated to G and let k be the number of 4-cliques in Γ. Recall that Proposition 1.1 gives $2b_2(G) - m_2(G) \leq h(G)$. We can view this lower bound as $b_2(G)$ plus the minimum dimension of the radical. Each 4-clique has 6 edges, so clearly $b_2(G)$ is equal to 6k minus the total number of shared edges in Γ. We will show that the realizing manifold M has $b_2(M)$ equal to $6k$ minus twice the number of possible dual sphere surgeries. Thus, to prove the theorem, we can show that the

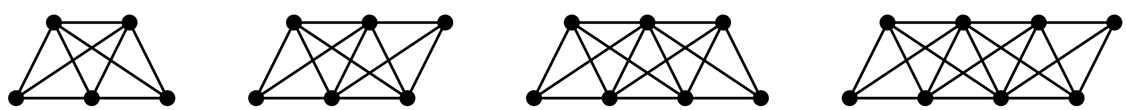
$$\text{(4)}\quad \#\text{ of shared edges} - \dim(\text{minimum radical}) = 2(\#\text{ possible dual 2-sphere surgeries}).$$

Fortunately, it suffices to show the above equation holds separately for each linear string of 4-cliques in the graph. That is, we can consider each row and each column of Γ separately. This is because the number of shared edges, number of basis elements of the minimum radical, and number of dual sphere surgeries in a single string of 4-cliques are additive and do not conflict with the count for other strings of 4-cliques in other rows and columns of Γ: the items counted in a horizontal string correspond only to vertical edges and vertical pairings of vertices in the string, and the items counted in a vertical string correspond only to horizontal edges and horizontal pairings of vertices in the string. Therefore, the separate counts will not conflict with each other. Further, Lemma 5.7 asserts that each of the dual sphere surgeries are possible when we consider the whole graph Γ.

However, in splitting up the proof we must take care to use the same choice of $\alpha \in H_4(G, \mathbb{Z}_2)$. Fortunately, we may assume that $\alpha = \alpha_1 + \ldots + \alpha_k$ ($c_i = 1$ for all i) minimizes the nullity. If instead $c_i = 0$ for some i so that $\alpha = \alpha_1 + \ldots + \alpha_{i-1} + \alpha_{i+1} + \ldots + \alpha_k$, the dimension of the radical can only increase. The generator α_i represents a choice of the ith 4-clique in the graph. Every 4-clique lies in exactly one row and one column. If the ith 4-clique is part of a string of an even number of 4-cliques (in either direction), the elements that would be part of the radical had c_i been 1 would no longer cause the form to be nondegenerate, so the count of the dimension of the radical will decrease by *at most* two. However, by construction of the graph, at least two edges in every 4-clique are not shared by any other 4-clique (the two diagonal edges). The unshared edges of the ith 4-clique are now basis elements of the radical. This causes the count of the dimension to increase by *at least* two.

Now, consider a string of ℓ connected 4-cliques. The number of shared edges is $\ell - 1$. As we saw in Example 4.3, there is an element of the radical if and only if ℓ is even. Thus for this string, the left-hand side of equation (4) is $\ell - 2$ if ℓ is even, and $\ell - 1$ if ℓ is odd. By Lemma 5.5 and Example 5.6, we can perform $\lfloor \frac{\ell-2}{2} \rfloor$ dual 2-sphere surgeries without changing π_1. Thus, the right-hand side of the equation is $\ell - 2$ if ℓ is even, and $\ell - 1$ if ℓ is odd. □

6.2. **4-cliques that share faces.** Next we will consider graphs of k 4-cliques that share *faces*, or triangles. First, consider the family of graphs represented by strings of k 4-cliques as exemplified by the graphs in Figure 10. In (a), $k = 2$; in (b), $k = 3$; in (c), $k = 4$; in (d), $k = 5$. We call a graph of this form a member of the String family.

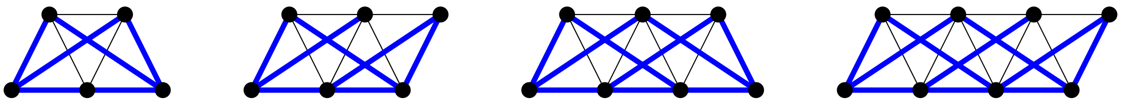


(a) (b) (c) (d)

FIGURE 10. Graphs in the String family

Theorem 6.2. *Let G be a RAAG whose defining graph is in the String family. For $k = b_4(G)$, then $h(G) = 3k + 6$ if k is even and $3k + 5$ if k is odd. In particular, G is cohomologically minimal.*

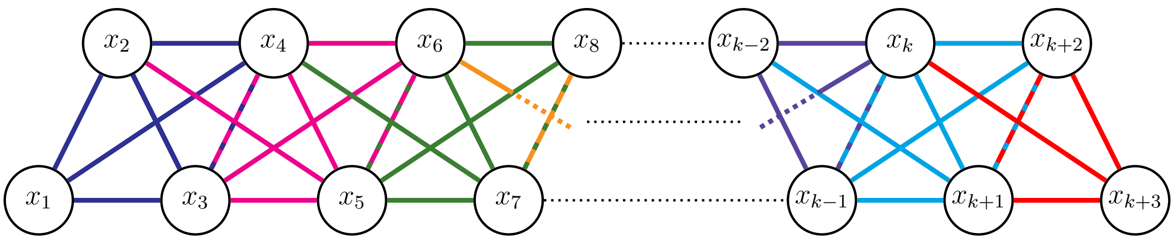
Proof. We will bound $h(G)$ below by calculating the dimension of the maximum isotropic subspace in terms of k. We will denote this dimension by d. Figure 11 **highlights** the edges of the graphs in Figure 10 which make up a maximum isotropic subspace in each case. When $k = 2$ (Figure 11 (a)), $d = 6$: two edges line the bottom of the graph, two edges are on either end of the string, and two are long diagonal edges. When $k = 3$ (Figure 11 (b)), $d = 7$: two edges line the bottom of the graph, two are end edges, and three are long diagonal edges. When $k = 4$ (Figure 11 (c)), $d = 9$: three edges line the bottom of the graph, two are end edges, and four are long diagonal edges. Following the pattern, we see that for general k, $d = (\lfloor \frac{k}{2} \rfloor + 1) + 2 + k$: $\lfloor \frac{k}{2} \rfloor + 1$ edges line the bottom of the graph, two are end edges, and k are long diagonal edges. Thus when k is even, $d = \frac{1}{2}(3k+6)$ and when k is odd, $d = \frac{1}{2}(3k+5)$. Twice the dimension d yields the necessary lower bound.



(a) (b) (c) (d)

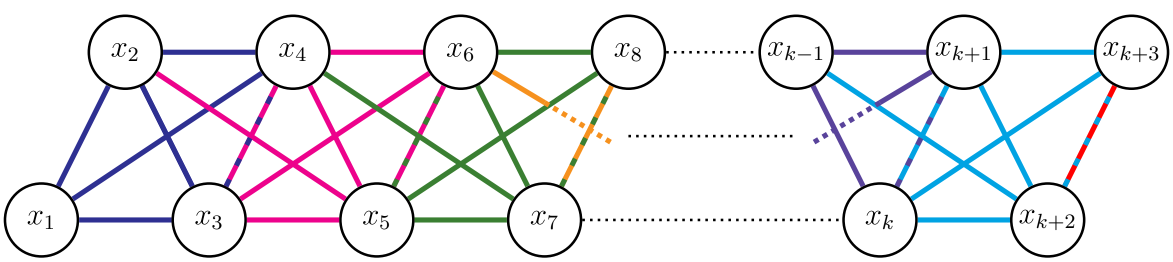
FIGURE 11. The **bold** edges of each graph form a maximal isotropic subspace.

To construct realizing 4-manifolds, we use 4-reductions applied to connected sums of $S^1 \times S^3$. If Γ has k 4-cliques, then it is not difficult to see that $b_1(G) = k+3$. We have two slightly different constructions distinguished by the parity of k. First consider the case when k is even. Begin with the connected sum of $k+3$ copies of $S^1 \times S^3$ in which $b_2 = 0$. Let $\{x_1, \ldots, x_{k+3}\}$ be the π_1 generators of each copy of S^1. Perform the following $(\frac{k}{2}+1)$ 4-reductions: $[x_1, x_2, x_3, x_4]$, $[x_2x_6, x_3, x_4, x_5]$, $[x_4x_8, x_5, x_6, x_7], \ldots, [x_{k-2}x_{k+2}, x_{k-1}, x_k, x_{k+1}]$, $[x_k, x_{k+1}, x_{k+2}, x_{k+3}]$, which are shown in the graph of π_1 below:



It is left to the reader to check that these 4-reductions yield all necessary relations for the correct π_1. Recall that each 4-reduction increases b_2 by 6. The 4-reductions result in a 4-manifold M with $\pi_1(M) = G$ and with $b_2(M) = 6(\frac{k}{2}+1) = 3k+6$, equal to the lower bound.

Now consider the case when k is odd. Again begin with the connected sum of $k+3$ copies of $S^1 \times S^3$, with the same π_1 generators $\{x_1, \ldots, x_{k+3}\}$. Perform the following $(\frac{k-1}{2}+1)$ 4-reductions: $[x_1, x_2, x_3, x_4]$, $[x_2x_6, x_3, x_4, x_5]$, $[x_4x_8, x_5, x_6, x_7], \ldots, [x_{k-1}x_{k+3}, x_k, x_{k+1}, x_{k+2}]$, as well as surgery to introduce the following commutator relation: $[x_{k+2}, x_{k+3}] = 1$. The relations created are shown in the graph below:



Each 4-reduction increases b_2 by 6 and the commutator surgery increases b_2 by 2. The result is a 4-manifold M with $\pi_1(M) = G$ and with $b_2(M) = 6(\frac{k-1}{2}+1)+2 = 3k+5$, equal to the lower bound. $\square$

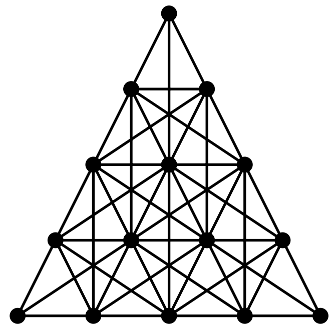


FIGURE 12. A graph in the Hex family

Next, consider the family of graphs with 4-cliques attached along faces whose vertices lie in a hexagonal grid (see Figure 12). In this setup, each triangle in the graph is not shared by more than three 3-cliques, and in each presentation of a 4-clique, the *long* edge is never a shared edge. We call graphs in this family *thick* if all boundary edges of the graph form an isotropic subspace. For example, Figure 13 shows two thick 4-cliques and a *thin* (not thick) 4-clique. We will call thick graphs lying in a hexagonal grid members of the Hex family.

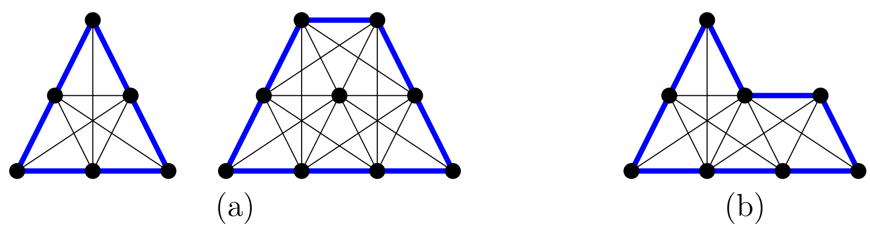


FIGURE 13. The graphs in (a) are *thick* and the graph in (b) is *thin*.

Theorem 6.3. *RAAGs with defining graphs belonging to the Hex family are cohomologically minimal.*

Proof. To prove this theorem, we will first identify which edges (of a graph in the Hex family) form an isotropic subspace of (1) in order to bound h from below, and then show this lower bound can be realized by a 4-manifold.

Consider an arbitrary graph in the Hex family. Since the graph is thick, all boundary edges form an isotropic subspace. Additionally, since every long edge of each 4-clique is not a shared edge, we can add it to the isotropic subspace. (We may do this because the shorter diagonal edges are not included in the isotropic subspace.) As an example, consider the graph in Figure 14 (a), which we will denote by Γ.

The boundary edges and the long diagonal edges of every 4-clique in Γ, **highlighted** in Figure 14 (b), form an isotropic subspace. Later, we will see that this isotropic subspace is maximal.

For any arbitrary graph in the Hex Family, we will construct a realizing 4-manifold as follows. Begin with the connected sum of b_1 copies of $S^1 \times S^3$. We will need to perform both 4-reductions on these generators as well as commutator surgeries in order for the 4-manifold

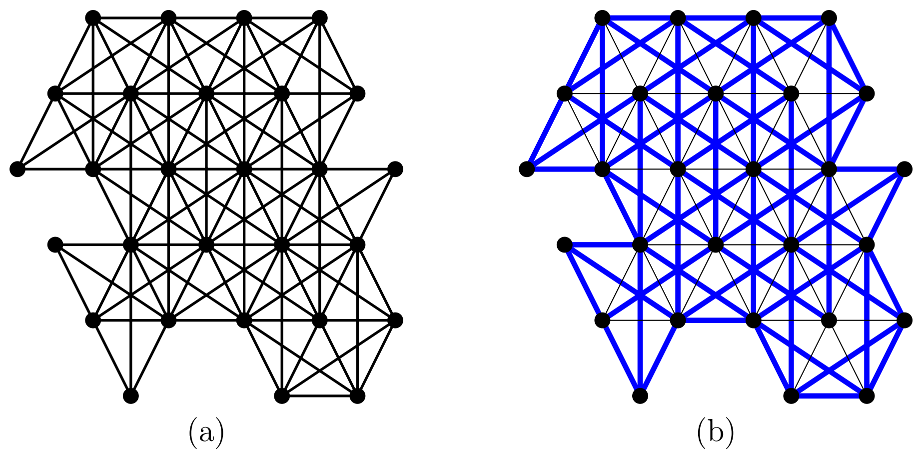


FIGURE 14. (a) A graph in the Hex family as well as (b) the edges that form an isotropic subspace

to have the correct π_1. Since it is intractable to give an arbitrary graph a set of generators and list the necessary 4-reductions and commutator surgeries, we will instead describe the pattern in which one can determine the surgeries from our example graph Γ. First note that all necessary 4-reductions will be of the form $[a, b, c, d]$, $[a, b, c, de]$, or $[a, b, c, def]$, where a, b, c, d, e, and f represent π_1 generators. In Section 5.3 we discussed the useful technique of shading a triangle in the graph bounded by the edges between a, b, and c. Figure 15 shows two possible yet equally sufficient constructions of a realizing 4-manifold M that has an associated π_1 graph Γ.

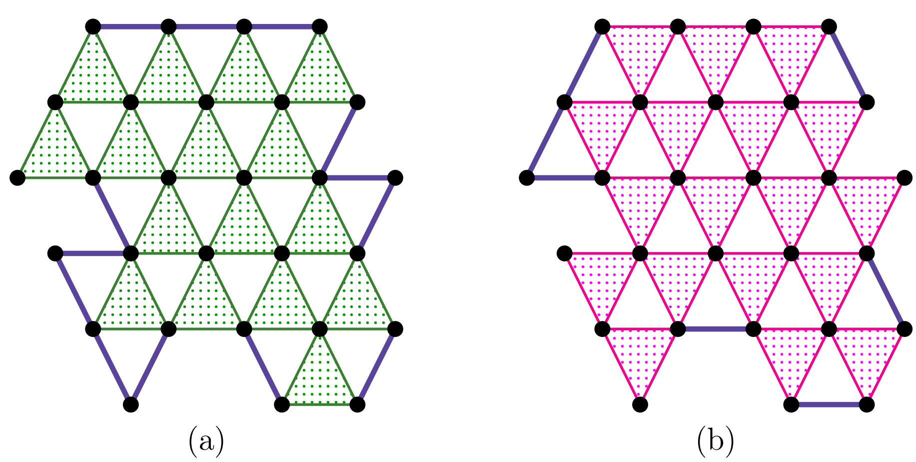
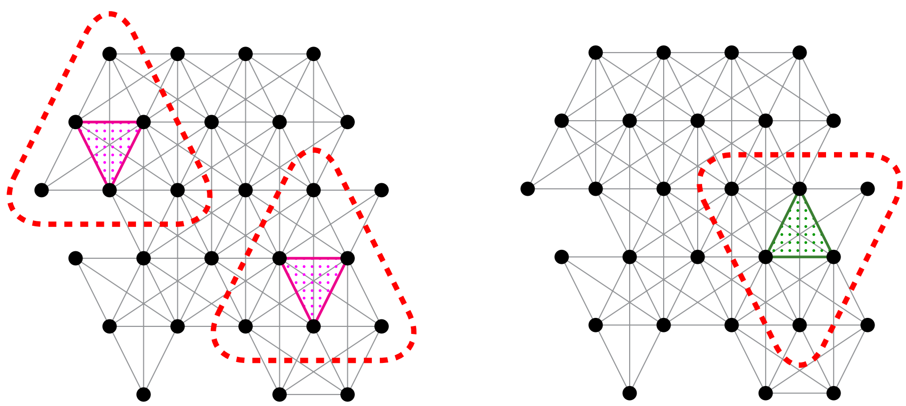


FIGURE 15. Two constructions for a realizing 4-manifold for a graph in the Hex family

In these constructions, the number of shaded triangles in the graph corresponds the number of necessary 4-reductions. The vertices on a particular shaded triangle correspond to three of the four elements of the 4-reduction. The fourth element of the 4-reduction is either another generator or a product of generators, depending on how many 4-cliques share the face of the shaded triangle. Each **bold** edge in Figure 15 corresponds to a necessary commutator surgery that will ensure the resulting 4-manifold will have the correct π_1. Note that all these **bold** edges are boundary edges of the graph which are not covered by any of the shaded triangles.

What remains to be seen is that this construction is "good enough." We will show that by following this pattern, we always construct a 4-manifold with the correct π_1 and with b_2 equal to twice the dimension of the described isotropic subspace. To do this, we will break down this pattern and show piece-by-piece that the cost of each 4-reduction and each surgery (in terms of adding b_2) can be balanced out by elements in the isotropic subspace. More specifically, we need only see that the cost x of each construction (in terms of adding b_2) can be balanced by finding half as many ($\frac{1}{2}x$) elements in the isotropic subspace.

To begin, let us discuss the costs of the three necessary types of 4-reductions: $[a, b, c, d]$, $[a, b, c, de]$, and $[a, b, c, def]$. First notice that any shaded triangle in the graph that is not along the boundary is created by a 4-reduction of the form $[a, b, c, def]$. In each case, the vertices representing a, b, and c are vertices of the shaded triangle, and the vertices representing d, e, and f are the fourth vertices of the three respective 4-cliques that share the shaded triangle. Examples of 4-reductions of the type $[a, b, c, def]$ can be viewed in Γ below:



Individually, each 4-reduction $[a, b, c, def]$ will eventually result in twelve commutator relations, those represented by the edges in Figure 16 (a). Automatically, the relations $[a, b] = 1$, $[a, c] = 1$, and $[b, c] = 1$ are created, represented by the edges in (b). The other surface-like relations (for example, $[a, def] = 1$) will be resolved by other 4-reductions and/or commutator surgeries. However, the three relations represented by the long diagonal edges highlighted in (c), are only introduced by this 4-reduction once the outer edges of (a) are created.

Each 4-reduction adds 6 to the total b_2 of the 4-manifold. This addition is balanced out by three edges that represent basis elements in the isotropic subspace. These three edges are the long diagonal edges of the three 4-cliques eventually created by this 4-reduction, shown in (c). Any remaining relations are introduced by other 4-reductions and/or commutator surgeries and their costs are balanced elsewhere.

Next, notice that any shaded triangle in the graph that has one edge along the boundary is created by a 4-reduction of the form $[a, b, c, de]$. As in the previous case, the vertices representing a, b, and c are vertices of the shaded triangle, and the vertices representing d

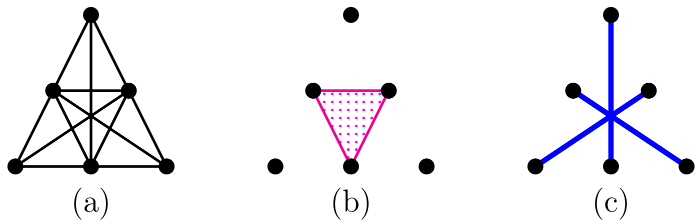
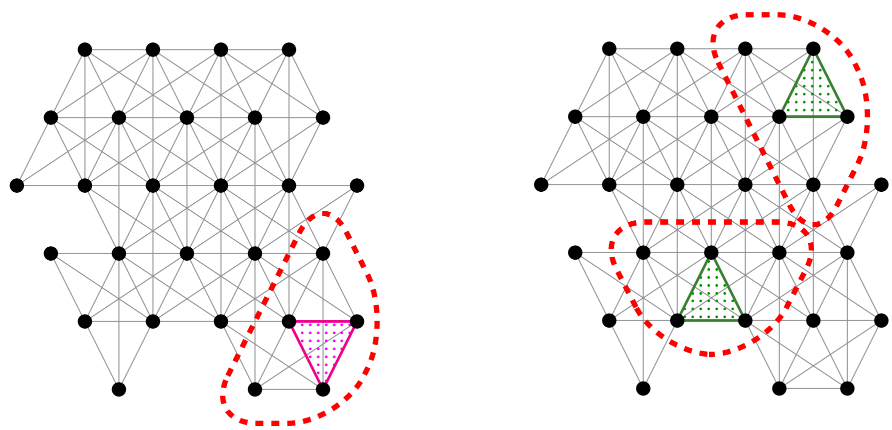


FIGURE 16. (a) The edges eventually created by a 4-reduction of type $[a, b, c, def]$, (b) the triangle created by a, b, and c, and (c) the edges belonging to the maximal isotropic subspace

and e are the fourth vertices of the two respective 4-cliques that share the shaded triangle. Examples of 4-reductions of the type $[a, b, c, de]$ can be viewed in Γ below:



Each 4-reduction of the form $[a, b, c, de]$ will eventually result in nine commutator relations, represented by the edges in Figure 17 (a). Again, we see the triangle in (b) represents the relations $[a, b] = 1$, $[a, c] = 1$, and $[b, c] = 1$. The two long diagonal edges in (c) will be resolved by other 4-reductions and/or commutator surgeries.

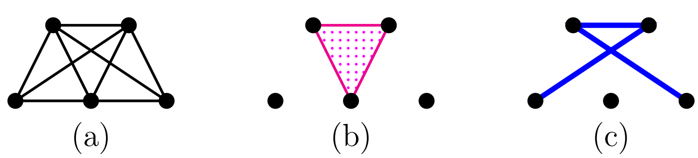
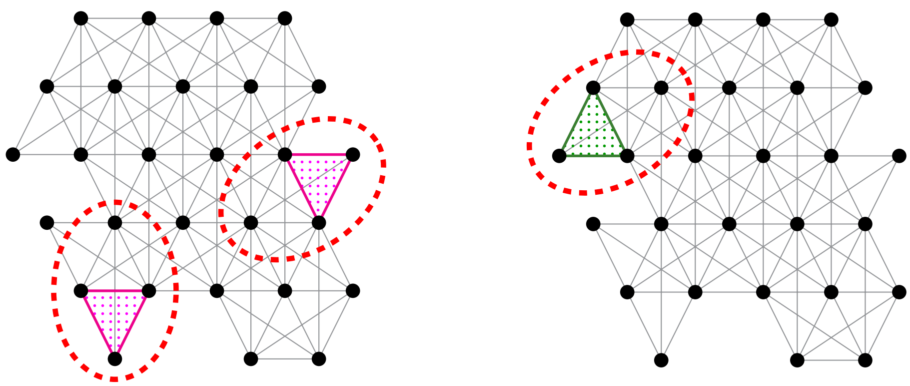


FIGURE 17. (a) The edges eventually created by a 4-reduction of type $[a, b, c, de]$, (b) the triangle created by a, b, and c, and (c) the edges belonging to the maximal isotropic subspace

Each 4-reduction of this form still adds 6 to the total b_2 of the 4-manifold. This addition is balanced out by three edges that represent basis elements in the isotropic subspace, the two long diagonal edges and the boundary edge in (c). Any remaining relations are introduced by other 4-reductions and/or commutator surgeries and their costs are balanced elsewhere.

The last type of 4-reduction, $[a, b, c, d]$, occurs when the shaded triangle includes two boundary edges. In this case, there is only one 4-clique in the graph containing the shaded triangle, the one formed by vertices representing a, b, c, and d. Examples of 4-reductions of this type can be viewed in Γ below:



Each 4-reduction of the form $[a, b, c, d]$ introduces 6 commutator relations, shown in Figure 18 (a) and adds 6 to the total b_2 of the 4-manifold. This addition is balanced out by three edges that represent basis elements in the isotropic subspace, the single long diagonal edge of the 4-clique and the two boundary edges shown in (c).

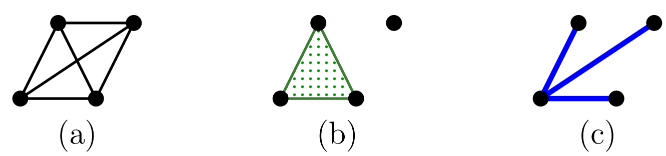


FIGURE 18. (a) The edges created by a 4-reduction of type $[a, b, c, d]$, (b) the triangle created by a, b, and c, and (c) the edges belonging to the maximal isotropic subspace

Lastly, we will consider the cost of the commutator surgeries. Each commutator surgery introduces a relation that represents a boundary edge of the graph, and the cost of the surgery (an addition of 2 to b_2) is balanced out by the fact that the corresponding boundary edge in the graph is in the isotropic subspace.

Since the cost of each 4-reduction and each surgery are balanced by elements in the isotropic subspace, it is clear that the pattern exemplified by Figure 14 (b) yields a maximum dimensional isotropic subspace and the construction pattern in Figure 15 yields a realizing 4-manifold. We remark that, interestingly, either pattern in Figure 15 is sufficient to construct a realizing manifold. $\square$

6.3. **RAAGs with nontrivial higher cohomology.** In graph theory, the *dimension* of a graph refers to the dimension of the largest clique in the graph. In terms of the cohomology of RAAGs, it is the cohomological dimension. Until now, we have only considered RAAGs up to dimension 4. There are many reasons for this.

RAAGs of dimension 4 are special, as 4 is the first dimension in which the cohomology ring really has an interesting influence on the possible values of $b_2(M)$ for arbitrary $M \in \mathcal{M}(G)$. Determining h is a delicate problem in groups of dimension 4 because calculations of m_2 as well as realizing manifold constructions are completely dependent on the ways in which 4-cliques interact in the graph. This provides evidence that the difficulty in determining the minimum b_2 problem of RAAGs lies in this dimension.

We now restrict the discussion to graphs of dimension k in which all $(k-1)$-cliques in the graphs are subgraphs of a k-clique. Let us say graphs under this restriction have *pure dimension k*. The next theorem gives a result for a family of cohomologically minimal RAAGs of pure dimension 5.

Theorem 6.4. *Let G be a RAAG with a defining graph containing k 5-cliques attached edge-to-edge as in Figure 19. Then $h(G) = 12k + 2$. In particular, G is cohomologically minimal.*

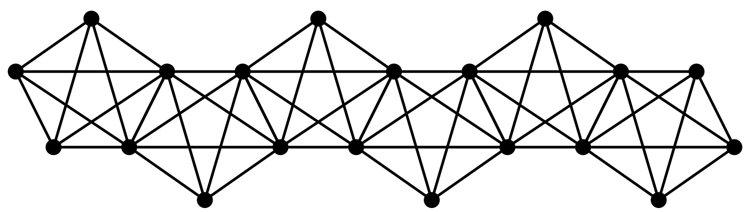


FIGURE 19. A graph of 5-cliques attached edge-to-edge

Proof. For large k, computing $m_2(G)$ is impractical; since $b_4(G) = 5k$, computing 2^{5k} ranks using a computer program is too time consuming. However, if we compute $m_2(G)$ for $k = 1, \ldots, 4$ we discover a pattern. The table below shows the calculations of the lower bound coming from the cohomology ring of G:

k	b_2	m_2	$2b_2 - m_2$
1	10	6	14
2	19	12	26
3	28	18	38
4	37	24	50
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$9k+1$	$6k$	$12k+2$

The calculation of b_2 is easy to see: each 5-clique has 10 edges, and $k-1$ edges of the graph are shared; therefore, $b_2 = 10k - (k-1) = 9k + 1$. Fortunately, there is a way to prove that the pattern for $m_2(G)$ continues for k larger than 4. To see that $m_2(G) = 6k$, one can find a sufficient lower bound for the dimension of the radical of (1). This will yield an upper bound for $m_2(G)$ and thus a lower bound for $h(G)$. In fact, we need only find a choice of $\alpha \in H_4(G; \mathbb{Z}_2)$ such that the dimension of the radical is $3k + 1$. If the dimension of the radical is bounded below by $3k + 1$, then $m_2(G)$ is bounded above by $6k$. Thus,

$2(9k+1) - 6k = 12k+2 \leq h(G)$. We will see that for each k, a 4-manifold M can be constructed with $b_2(M) = 12k+2$, which will guarantee that $m_2(G)$ is also bounded below by (and thus equal to) $6k$.

A graph G with k 5-cliques attached edge-to-edge will have $3k+2$ vertices, $\{s_1, \ldots, s_{3k+2}\}$. We can label the vertices in a graph as shown in Figure 20. Consider the following ordering

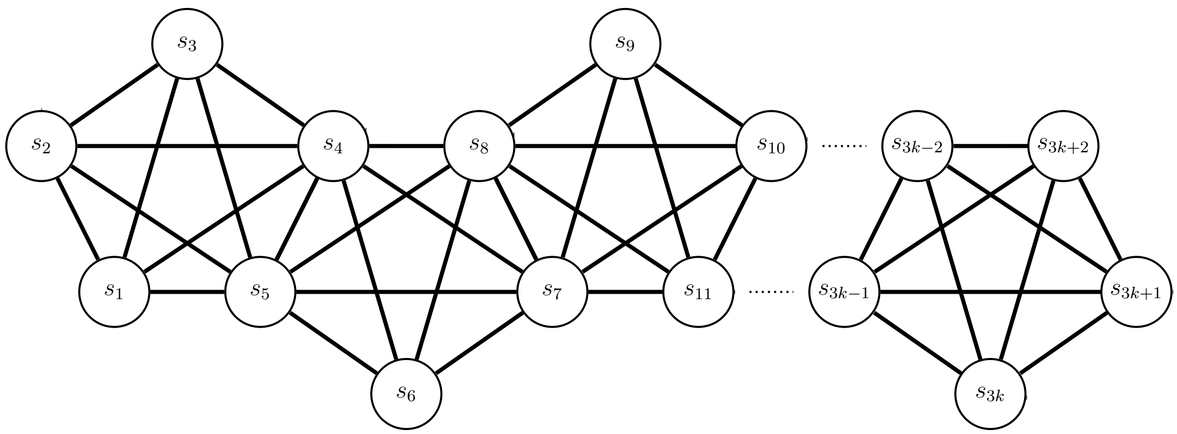


FIGURE 20. A graph of k 5-cliques attached edge-to-edge

of the basis elements for $H^2(G; \mathbb{Z}_2)$:

$$\{z_{12},\ z_{13},\ z_{14},\ z_{15},\ z_{23},\ z_{24},\ z_{25},\ z_{34},\ z_{35},\ z_{45},\ z_{46},\ z_{47}, \ldots,\ z_{(3k+1)(3k+2)}\}$$

For a choice of $\alpha \in H_4(G; \mathbb{Z}_2)$, we can find a basis for the radical of (1), as we did in Example 4.2:

If $k = 1$ and $\alpha = s_{1234} + s_{2345}$ then the following 4 elements form a basis for the radical: $\{z_{12} + z_{25},\ z_{13} + z_{35},\ z_{14} + z_{45},\ z_{15}\}$. One can see this by verifying that each element cupped with an arbitrary generator $z \in H^2(G; \mathbb{Z}_2)$ and evaluated on the class α is 0 mod 2. For example, $\langle (z_{12} + z_{25}) \cup z, s_{1234} + s_{2345} \rangle = \langle z_{12} \cup z, s_{1234} \rangle + \langle z_{25} \cup z, s_{2345} \rangle$ is equal to 0 for all $z \neq z_{34}$ and equal to 0 mod 2 for $z = z_{34}$. The edges represented by the radical's basis[2] are highlighted in Figure 21.

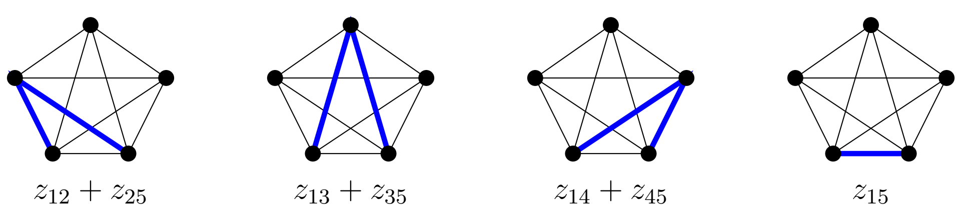
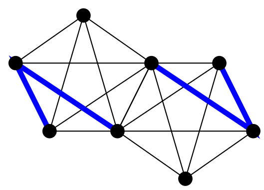


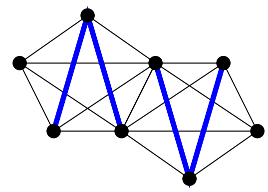
FIGURE 21. 4 basis elements in the radical for $k = 1$ and $\alpha = s_{1234} + s_{2345}$

[2]In the case that a basis element is a sum of generators of $H^2(G; \mathbb{Z}_2)$, the edges of each generator in the summand are **highlighted** in the graph.

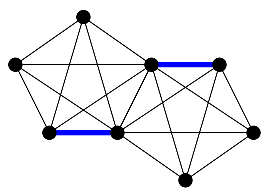
If $k = 2$ and $\alpha = s_{1234} + s_{2345} + s_{4567} + s_{5678}$, then the following 7 elements form a basis for the radical: $\{z_{12} + z_{25},\ z_{13} + z_{35},\ z_{14} + z_{45} + z_{58},\ z_{15},\ z_{46} + z_{68},\ z_{47} + z_{78},\ z_{48}\}$. The corresponding edges are **highlighted** in Figure 22.



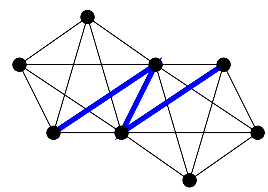
$z_{12} + z_{25}$ and $z_{47} + z_{78}$



$z_{13} + z_{35}$ and $z_{46} + z_{68}$



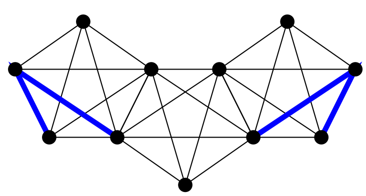
z_{15} and z_{48}



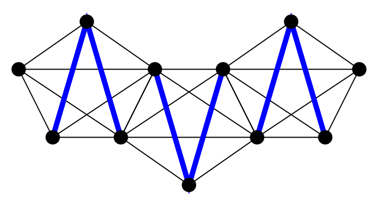
$z_{14} + z_{45} + z_{58}$

FIGURE 22. 7 basis elements in the radical for $k = 2$ and $\alpha = s_{1234} + s_{2345} + s_{4567} + s_{5678}$

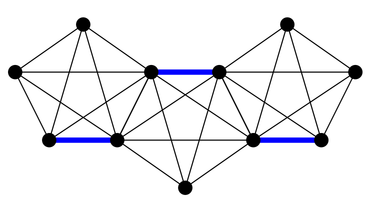
If $k = 3$ and $\alpha = s_{1234} + s_{2345} + s_{4567} + s_{5678} + s_{789(10)} + s_{89(10)(11)}$, then the following 10 elements give a basis for the radical: $\{z_{12} + z_{25},\ z_{13} + z_{35},\ z_{14} + z_{45} + z_{58},\ z_{15},\ z_{46} + z_{68},\ z_{47} + z_{78} + z_{8(11)},\ z_{48},\ z_{79} + z_{9(11)},\ z_{7(10)} + z_{(10)(11)},\ z_{7(11)}\}$. The corresponding edges are **highlighted** in Figure 23.



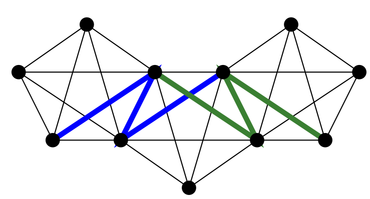
$z_{12} + z_{25}$ and $z_{7(10)} + z_{(10)(11)}$



$z_{13} + z_{35}$, $z_{46} + z_{68}$, and $z_{79} + z_{9(11)}$



z_{15}, z_{48}, and $z_{7(11)}$



$z_{14} + z_{45} + z_{58}$ and $z_{47} + z_{78} + z_{8(11)}$

FIGURE 23. 10 basis elements in the radical for $k = 3$ and $\alpha = s_{1234} + s_{2345} + s_{4567} + s_{5678} + s_{789(10)} + s_{89(10)(11)}$

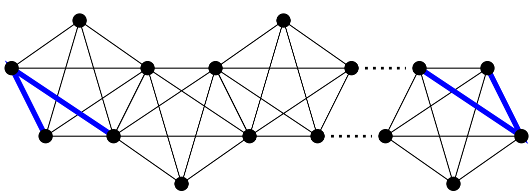
We are developing a pattern to determine a basis for the radical of (1) for any k. First, we order the basis elements of $H_4(G;\mathbb{Z}_2)$ as follows:

$$\{s_{1234},\ s_{1235},\ s_{1245},\ s_{1345},\ s_{2345},\ s_{4567},\ s_{4568},\ s_{4578},\ s_{4678},\ s_{5678},\ldots,\ s_{(3k-1)(3k)(3k+1)(3k+2)}\}$$

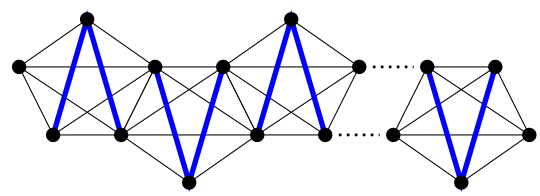
Note that each 5-clique has five basis elements in $H_4(G;\mathbb{Z}_2)$, ordered consecutively in the set above. Consider the following choice of α, in which only the first and last basis elements of each 5-clique are nonzero:

$$\alpha = s_{1234} + s_{2345} + s_{4567} + s_{5678} + \ldots + s_{(3k-2)(3k-1)(3k)(3k+1)} + s_{(3k-1)(3k)(3k+1)(3k+2)}.$$

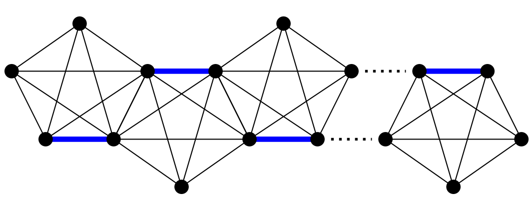
Notice that this choice of α agrees with the previous choices for small k. Based on the developed pattern, we can find a basis for the radical for any k:



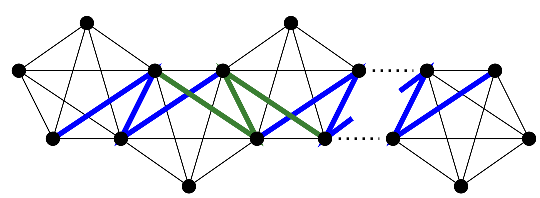
$z_{12} + z_{25}$ and $z_{(3k-2)(3k+1)} + z_{(3k+1)(3k+2)}$



$z_{13} + z_{35}$, $z_{46} + z_{68}$, $z_{79} + z_{9(11)}$, $\ldots$, and $z_{(3k-2)(3k)} + z_{(3k)(3k+2)}$



z_{15}, z_{48}, $z_{7(11)}$, $\ldots$, and $z_{(3k-2)(3k+2)}$



$z_{14} + z_{45} + z_{58}$, $z_{47} + z_{78} + z_{8(11)}$,
$z_{7(10)} + z_{(10)(11)} + z_{(11)(14)}$, $\ldots$, and
$z_{(3k-5)(3k-2)} + z_{(3k-2)(3k-1)} + z_{(3k-1)(3k+2)}$

Counting the edges **highlighted** above, we conclude there are $2 + k + k + (k-1) = 3k + 1$ elements in this basis for the radical. As previously noted, this implies the rank of the form (1) for our choice of α is $6k$.

The realizing manifold construction for the upper bound is quite straightforward. We start with two copies of a 4-torus and $k-1$ copies of $T^2 \times \Sigma_2$. The required surgeries are most easily explained with an example. Let $k = 4$. Let π_1 of the two 4-tori be generated by $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$, and let π_1 of the three copies of $T^2 \times \Sigma_2$ be generated by $\{a_1, a_2\}$ and $\{b_1, b_2, b_3, b_4\}$, $\{c_1, c_2\}$ and $\{d_1, d_2, d_3, d_4\}$, and $\{t_1, t_2\}$ and $\{s_1, s_2, s_3, s_4\}$. (See Figure 24 for the graphical representations of π_1.) Before surgeries, $b_2(\#2T^4\#3(T^2 \times \Sigma_2)) = 42$.

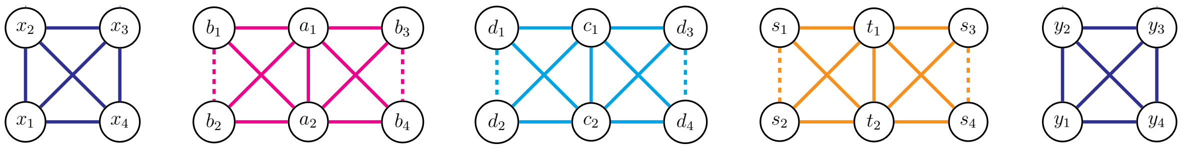


FIGURE 24. A graph representing π_1 of the products of surfaces necessary for this 4-manifold construction

Perform surgery on the connected sum to create the following 12 identifications:

$$\begin{array}{lll} x_1 = b_2 & b_3 = c_1 & d_4 = t_2 \\ x_3 = b_1 & b_4 = d_2 & y_1 = s_4 \\ x_4 = a_2 & c_2 = s_2 & y_2 = t_1 \\ a_1 = d_1 & d_3 = s_1 & y_3 = s_3 \end{array}$$

These do not change b_2. Lastly, perform surgeries to introduce the following four commutator relations: $[x_2, a_1] = 1$, $[x_4, c_2] = 1$, $[b_3, y_2] = 1$, and $[d_4, y_4] = 1$. After these surgeries, $b_2 = 42 + 4(2) = 50$. The resulting graph associated to π_1 of this realizing manifold is shown in Figure 25.

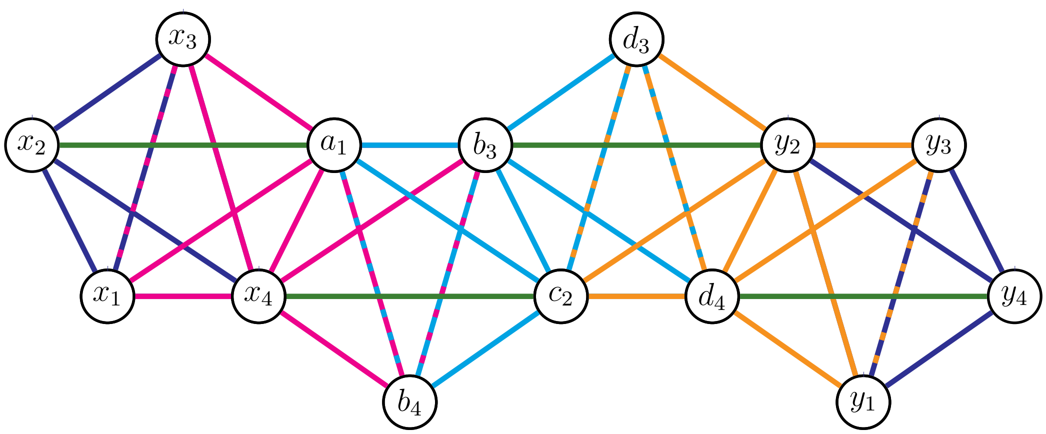


FIGURE 25. A graph representing π_1 of a realizing manifold for four 5-cliques attached edge-to-edge

For a graph with k 5-cliques, $k-1$ copies of $T^2 \times \Sigma_2$ are required, and the necessary identification surgeries follow the pattern described by the example. Each copy of T^4 adds 6 to the count of b_2, and each copy of $T^2 \times \Sigma_2$ adds 10. Lastly, k commutator surgeries are necessary and each increase b_2 by 2. The resulting 4-manifold M has $b_2(M) = 6(2) + 10(k-1) + 2k = 12k + 2$. □

Recall Proposition 4.4 which states that $m_2(G)$ is even for a RAAG G. We use this proposition in the proofs of the next two theorems.

Theorem 6.5. *Let G be a RAAG with a defining graph containing k 6-cliques attached edge-to-edge as in Figure 26. Then $h(G) = 14k + 2$.*

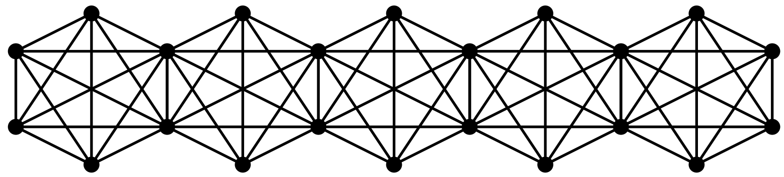


FIGURE 26. A graph of 6-cliques attached edge-to-edge

Proof. Let G be a RAAG with a defining graph of k 6-cliques attached edge-to-edge as in Figure 26. Each 6-clique has 15 edges and $k-1$ edges in the graph are shared, so $b_2(G) = 15k - (k-1) = 14k + 1$. Because $b_2(G)$ is odd, we know that $14k + 2 \leq h(G)$ by Proposition 4.4.

We begin the construction of a realizing manifold for $h(G)$ with the connected sum of $k-1$ copies of $T^2 \times \Sigma_3$, one copy of $T^2 \times \Sigma_2$ and one 4-torus. As in the proof of Theorem 6.4, we will see the pattern of necessary identification surgeries with an example. Let $k = 3$. Start with $(T^2 \times \Sigma_2)\#(T^2 \times \Sigma_3)\#(T^2 \times \Sigma_3)\#T^4$, where $\pi_1(T^2 \times \Sigma_2)$ is generated by $\{x_1, x_2\}$ and $\{y_1, y_2, y_3, y_4\}$ and $\pi_1(T^4)$ is generated by $\{u_1, u_2, u_3, u_4\}$. Let $\{s_1, s_2\}$ and $\{t_1, \ldots, t_6\}$ generate π_1 of the first copy of $T^2 \times \Sigma_3$ and let $\{w_1, w_2\}$ and $\{z_1, \ldots, z_6\}$ generate π_1 of the second. Figure 27 shows a graphical representation of the fundamental group of each summand of the 4-manifold.

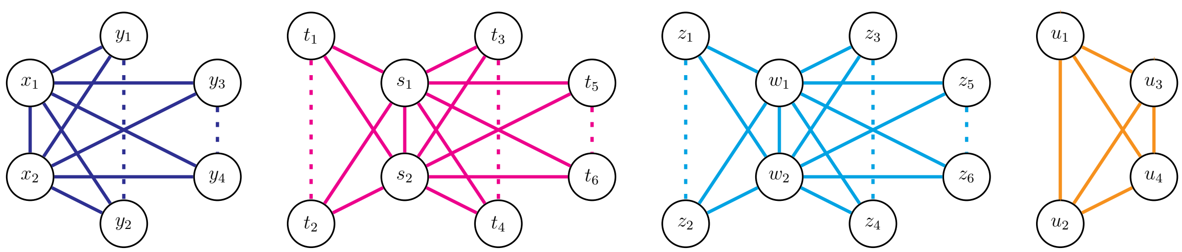


FIGURE 27. A graph representing π_1 of the products of surfaces necessary for this 4-manifold construction

Perform surgeries to create the following identifications:

$$\begin{array}{lll} y_1 = t_1 & t_3 = z_1 & z_3 = u_1 \\ y_2 = t_2 & t_4 = z_2 & z_4 = u_2 \\ y_3 = s_1 & t_5 = w_1 & z_5 = u_3 \\ y_4 = s_2 & t_6 = w_2 & z_6 = u_4 \end{array}$$

These surgeries yield a 4-manifold with the correct fundamental group. (See Figure 28.) This example shows the identification surgery pattern one would use to construct a realizing manifold for any k. The copy of $T^2 \times \Sigma_2$ adds 10 to the count of b_2, each copy of $T^2 \times \Sigma_3$ adds 14, and the 4-torus adds 6. The resulting manifold M has $b_2(M) = 10(1) + 14(k-1) + 6 = 14k + 2$. $\square$

The last family of RAAGs we will explore in this paper is a family of graphs of pure dimension 7.

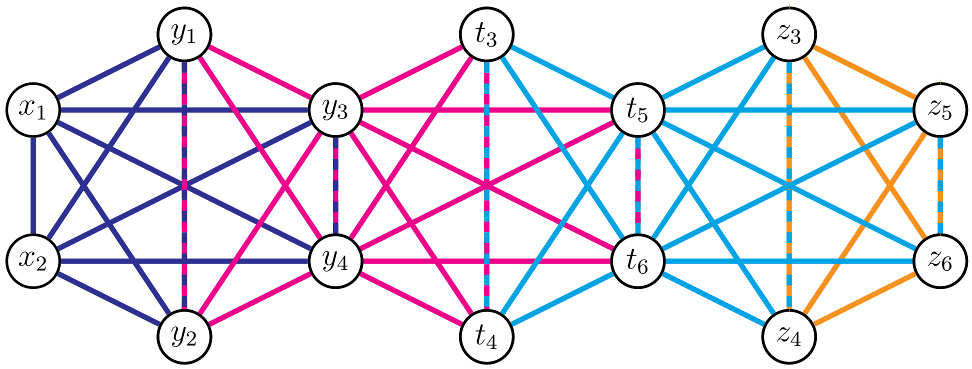


FIGURE 28. A graph representing π_1 of a realizing manifold for three 6-cliques attached edge-to-edge

Theorem 6.6. *Let G be a RAAG with a defining graph containing k 7-cliques attached edge-to-edge as in Figure 29. Then $h(G) = 20k + 2$.*

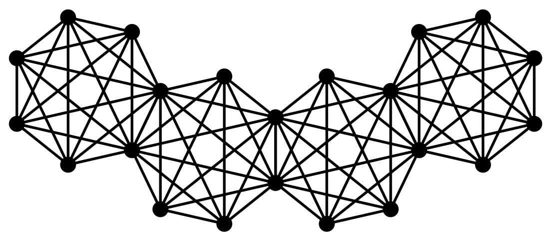


FIGURE 29. A graph of 7-cliques attached edge-to-edge

Proof. Let G be a RAAG with a defining graph of k 7-cliques attached edge-to-edge, as in Figure 29. Each 7-clique has 21 edges and $k-1$ edges in the graph are shared, so $b_2(G) = 21k - (k-1) = 20k + 1$. Because $b_2(G)$ is odd, Proposition 4.4 asserts that $20k + 2 \leq h(G)$.

The following construction of a realizing manifold for $h(G)$ contains $k-1$ copies of $T^2 \times \Sigma_3$ as well as one copy of $T^2 \times \Sigma_2$ and one 4-torus.

As in the proofs of Theorems 6.4 and 6.5, we will see the pattern of necessary identification surgeries with an example. Let $k = 3$. Start with $T^4 \#(T^2 \times \Sigma_3)\#(T^2 \times \Sigma_3)\#(T^2 \times \Sigma_2)$. Let $\{x_1, x_2\}$ and $\{y_1, \ldots, y_6\}$ generate π_1 of the first copy of $T^2 \times \Sigma_3$, and let $\{s_1, s_2\}$ and $\{t_1, \ldots, t_6\}$ generate π_1 of the second. Let $\{z_1, z_2, z_3, z_4\}$ generate $\pi_1(T^4)$ and let $\{u_1, u_2\}$ and $\{v_1, v_2, v_3, v_4\}$ generate $\pi_1(T^2 \times \Sigma_2)$. See Figure 30 for the graphical representations of π_1 of each summand of the 4-manifold.

Perform surgeries to create the following identifications:

$$\begin{array}{lll} y_1 = z_3 & x_1 = t_3 & s_1 = v_3 \\ y_2 = z_4 & y_5 = t_1 & t_5 = v_1 \\ y_3 = z_1 & y_6 = t_2 & t_6 = v_2 \end{array}$$

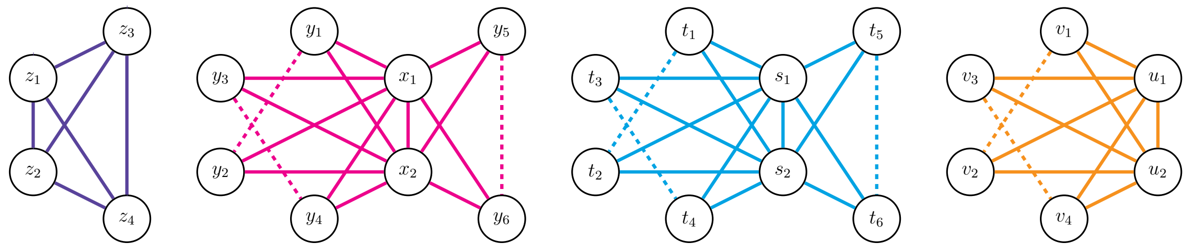


FIGURE 30. A graph representing π_1 of the products of surfaces necessary for this 4-manifold construction

These surgeries do not change b_2. Next perform the following three 4-reductions:

$$[y_4, z_2, x_1y_1, x_2y_2], \ [x_2, t_4, y_5s_1, y_6, s_2], \ [s_2, v_4, t_5u_1, t_6u_2]$$

These 4-reductions result in a 4-manifold with the correct π_1. (See Figure 31.) This example shows the pattern one would use to construct a realizing manifold for any k. The copy of $T^2 \times \Sigma_2$ adds 10 to the count of b_2, each copy of $T^2 \times \Sigma_3$ adds 14, and the 4-torus adds 6. In addition, k 4-reductions are required and each adds 6 to b_2. The resulting manifold M has $b_2(M) = 6 + 14(k-1) + 10 + 6k = 20k + 2$. □

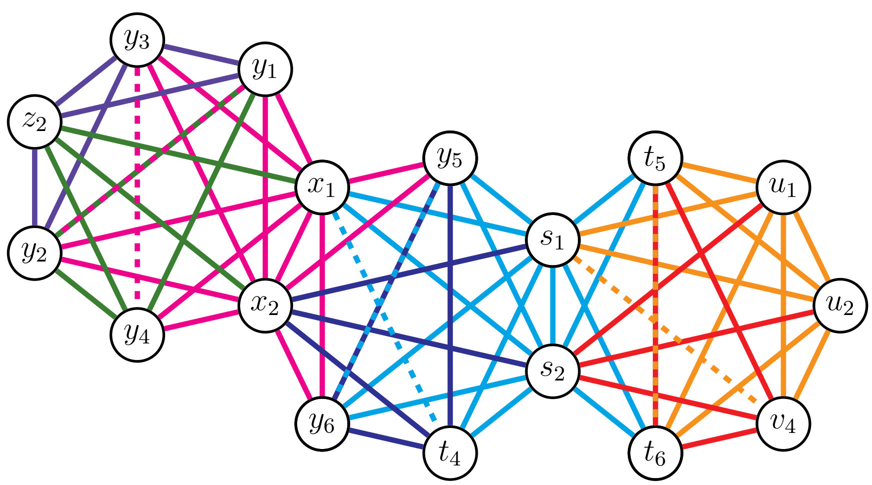


FIGURE 31. A graph representing π_1 of a realizing manifold for three 7-cliques attached edge-to-edge

7. Concluding Remarks

The author knows no examples of RAAGs that are *not* cohomologically minimal. We therefore make the following conjecture that is stated in the introduction:

Conjecture 1.6. *All RAAGs are cohomologically minimal. That is, if G is a RAAG, $h(G) = 2b_2(G) - m_2(G)$.*

Remark. This conjecture does not hold for all finitely presented groups. Consider the following counterexample. Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. A classifying space for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is $\mathbb{RP}^\infty \times \mathbb{RP}^\infty$. Using the Universal Coefficient Theorem, the Künneth formula for homology, and the homology of $\mathbb{RP}^\infty$,we see that $b_i(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = 0$ for $i = 1, 2$ and 1 for $i = 0$. A realizing 4-manifold for $h(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ is constructed in [8] from $(L(2,1) \times S^1)\#(S^1 \times S^3)$. Surgery is then performed to identify the generator of $\pi_1(L(2,1))$ and the generator of $\pi_1(S^1 \times S^3)$. Let a be the generator of $\pi_1(S^1)$ from $L(2,1) \times S^1$. Surgery is performed on a^2, which results in a 4-manifold with the correct π_1 and $b_2 = 0$. However, $H^*(\mathbb{RP}^\infty \times \mathbb{RP}^\infty; \mathbb{Z}_2)$ is just the polynomial ring $\mathbb{Z}_2[\alpha, \beta]$. Thus, the form (1) must be nondegenerate and so $m_2(\mathbb{RP}^\infty \times \mathbb{RP}^\infty)$ will be positive. Then $2b_2(G) - m_2(G) < h(G) = 0$.

More generally, the author suspects that the tools described in Section 5 will be sufficient for all constructions of realizing manifolds for RAAGs. If true, this would mean that all such realizing manifolds have zero signature, as in the cases of free and free abelian groups [9].

The greatest obstacle in proving this conjecture is generalizing current results without using induction. One may expect to find an inductive way to calculate h. For example, given any two subgraphs Γ_1 and Γ_2 of Γ, one may expect there is a relationship between $h(G_1)+h(G_2)$ and $h(G)$, as is found in the free abelian case [8, Theorems 8,9]. Unfortunately, for general RAAGs, this is not the case; it is not guaranteed that a realizing manifold for $h(G)$ can be constructed from realizing 4-manifolds for $h(G_1)$ and $h(G_2)$.

In general, it is difficult to have inductive results involving graphs, although Theorems 1.3 and 1.4 provide a good beginning. It is not clear whether we should induct on vertices or edges. Because vertices alone correspond only to b_1, adding vertices without adding edges has no effect on computing h. However, adding edges can change the value of h drastically. If the added edge does not form a new 4-clique in the graph, we know from Theorem 1.5 that this increases h by 2. However, if adding an edge creates additional 4-cliques in the graph, the change in h depends on the structure of the graph. In fact, adding one edge in the graph may result in an entirely different construction of a new realizing manifold.

In light of this difficulty, the only known examples of cohomologically minimal RAAGs belong to infinite families of graphs in which induction on patterns allows us to calculate h for all groups in the family. Beyond finding new families of graphs, however, it is unclear how to proceed in proving this conjecture.

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